The Gromov-Lawson obstruction and the Geroch conjecture

Luke Volk

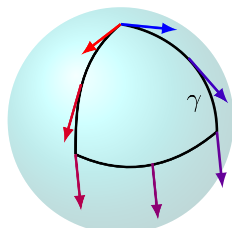
First up, a little review a little about curvature (Riemann, Ricci, and scalar), and then we move on to reviewing some spin geometry. Finally, we conclude with Gromov and Lawson's proof of the Geroch conjecture: T^n does not admit a metric with positive scalar curvature. Sections 2 and 3 in particular closely follow Lawson & Michelsohn's textbook on spin geometry [2] (it's very good, you should read it—I don't do it justice recounting the ideas here).

Contents

1 Curvature

1.1 The Riemann curvature tensor

In $\mathbb{R}^n$, we may draw any polygon with a vertex p and note that parallel transporting a vector $v \in T_p\mathbb{R}^n$ around the loop will yield v. That is, dragging v around $\mathbb{R}^n$ and returning back to p leaves the vector unchanged. However, parallel transport is not always as ideal on other manifolds. Consider S^2 and a triangle γ on S^2 which traces the boundary of an octant of S^2. It is easily checked visually that parallel transport of a vector around γ does not preserve the vector.



The difference between $\mathbb{R}^n$ and S^2 in this scenario amounts to the presence of "curvature" on S^2, whereas $\mathbb{R}^n$ is flat. The infinitesimal analogue for this is determining whether ∇_v and ∇_w commute for parallel sections v and w. That is, the operator

$$[\nabla_v, \nabla_w] = \nabla_v\nabla_w - \nabla_w\nabla_v$$

measures curvature to some degree. We define the **Riemann curvature** on a manifold X to be $R(v, w) \in \operatorname{End}\Gamma(TX)$ by:

$$R(v, w) := [\nabla_v, \nabla_w] - \nabla_{[v,w]}.$$

A succinct way of defining the curvature is by assigning $R(v, w)$ as the degree to which $\nabla\colon v \mapsto \nabla_v$ fails to be a Lie algebra homomorphism.

Generalizing to a connection $\nabla\colon \Gamma(TX) \times \Gamma(E) \to \Gamma(E)$ on a vector bundle $E \to X$, the **Riemann curvature** of E is given analogously as the map $R\colon \Gamma(TX) \times \Gamma(TX) \times \Gamma(E) \to \Gamma(E)$ given by:

$$R(v, w)\sigma := [\nabla_v, \nabla_w]\sigma - \nabla_{[v,w]}\sigma.$$

We sometimes fix v, w and write the Riemann curvature tensor as a map $R_{v,w}\colon \Gamma(E) \to \Gamma(E)$.

1.2 Ricci curvature

Because the Riemann curvature tensor is a complicated object, geometers often turned their attention to simpler notions of curvature. One important notion derived from the Riemann curvature tensor is the Ricci curvature tensor. The **Ricci curvature** at $x \in X$ is the map $\operatorname{Ric}_x\colon T_pX \times T_pX \to \mathbb{R}$ defined by

$$\operatorname{Ric}_x(v, w) = \operatorname{tr}(u \mapsto R(u, v)w).$$

This trace is in some sense an average of the Riemann curvature tensor. As someone described it on math.stackexchange:

For any unit vector $v \in T_p X$, $\mathrm{Ric}_x(v,v)$ is the sum of the sectional curvatures of planes spanned by v and other elements of an orthonormal basis. Since it is symmetric and bilinear, it is completely determined by $\mathrm{Ric}_x(v,v)$ for unit vectors v.

1.3 Scalar curvature

Now suppose that we are given a Riemannian metric g on X. By non-degeneracy, there is a unique linear map $L\colon T_x X \to T_x X$ such that for all $v, w \in T_x X$:

$$\mathrm{Ric}_x(v,w) = g(L(v), w).$$

We define the **scalar curvature** $Scal\colon X \to \mathbb{R}$ to be:

$$Scal(x) \overset{\text{def}}{=} \mathrm{tr}_g \mathrm{Ric}_x = \mathrm{tr}(L) = \sum_j \mathrm{Ric}_x(e_j, e_j) = \sum_{i,j} g(R(e_i, e_j)e_j, e_i),$$

where $e_1, \ldots, e_n$ is an orthonormal basis for $T_x X$. It is in some sense a doubly averaged curvature, but can also be seen as being related to the deviation of the volume of a small geodesic ball on M from the volume of the standard ball in Euclidean space by noting that:

$$\frac{\mathrm{Vol}(B(p;\varepsilon) \subset M)}{\mathrm{Vol}(B(0;\varepsilon) \subset \mathbb{R}^n)} = 1 - \frac{Scal}{6(n+2)}\varepsilon^2 + O(\varepsilon^4). \qquad (\star)$$

1.4 Difficulties with scalar curvature

What makes scalar curvature interesting is perhaps its deceiving simple nature. As a function $Scal\colon X \to \mathbb{R}$, the scalar curvature seems like a simple notion of curvature, not taking the form of a tensor. In two dimensions, the scalar curvature is especially simple.

Observation 1. *In two dimensions, the scalar curvature is exactly twice the Gaussian curvature.*

Because of this close relation to the Gaussian curvature in dimension two, the scalar curvature has a lot of topological significance (e.g. through the Gauss-Bonnet theorem), the sign telling you for example that the only closed surfaces with metrics of positive scalar curvature are those with positive Euler characteristic, both of which do not admit a metric with scalar curvature $\leqq 0$. In particular,

Observation 2. *The 2-torus does not admit a metric with positive scalar curvature.*

It's instructive to note Kazdan-Warner's 1975 result where they described which smooth functions arise as the scalar curvature of metrics on a closed manifold X of dimension at least 3. They proved that the manifold must satisfy one of the following three cases:

1. Every $f\colon X \to \mathbb{R}$ is the scalar curvature of some metric on X.
2. $f\colon X \to \mathbb{R}$ is the scalar curvature of some metric on X $\iff$ f is zero or negative somewhere.
3. $f\colon X \to \mathbb{R}$ is the scalar curvature of some metric on M $\iff$ f is negative somewhere.

The conclusion is that any such manifold has a metric with negative scalar curvature (moreover can be chosen so it is *constant* negative scalar curvature!).

The generalized conjecture that the n-torus does not admit a metric with positive scalar curvature is the so-called **Geroch conjecture**, and is considerably more difficult than the $n = 2$ case seems to suggest. In higher dimensions, Gauss-Bonnet does not explicitly feature the scalar curvature, the $n = 2$ case being something of a fluke. The conjecture for $n \leqq 7$ was proved by Schoen and Yau in the late 70s.

So why the apparent difficulty surrounding the $n \geqq 3$ case? Larry Guth [1] suggests that the definition ($\star$) does not actually pose much use, only telling us a limiting behaviour of small balls. Guth points out that in the case of Ricci curvature we have the **Bishop-Gromov inequality**:

Proposition 1.1 (Bishop-Gromov inequality). *If (X, g) is a Riemannian n-manifold with non-negative Ricci curvature, then for any $x \in X$ and any radius r:*

$$Vol(B(x; r) \subset X) \leqq Vol(B(0; 1) \subset \mathbb{R}^n) r^n.$$

This inequality is notably stronger than ($\star$), being a global inequality that describes balls of any size, and thus we have a better chance of a topological description of X popping out of the Ricci curvature than out of the scalar curvature which merely provides us with a very localized condition for balls.

Guth remarks that geometers proved global geometric inequalities for manifolds with non-negative Ricci or sectional curvature in the 30s, 40s, and 50s, appearing relatively soon after they were pursued. On the flip side, a global geometric inequality for metrics with non-negative scalar curvature was not proven until the late 1970s.

The key idea in Schoen and Yau's proof of the Geroch conjecture (which debuted in 1978) for $n \leqq 7$ was that if (X, g) has positive scalar curvature and $\Sigma \subset X$ is a stable minimal hypersurface, then on average, Σ has positive scalar curvature, too. Note that in particular this seems promising for understanding the Geroch conjecture for $n = 3$ as the (hyper)surface $\Sigma \subset T^3$ would be a 2-manifold with positive scalar curvature, and the scalar curvature of 2-manifolds is much better understood due to the connection with the Gaussian curvature!

A minimal surface (one with zero mean curvature, i.e. area minimizing) is **stable** if there are no directions in which the area increases.

1.5 What we want to show

So now given that we are convinced to some extent that the scalar curvature is mysterious and difficult, we will show the proof of the Geroch conjecture given by Gromov and Lawson in the early 80s using index theory!

2 Spin geometry

2.1 Spin manifolds and spinor bundles

We are first going to dive in to a little **spin geometry**, maybe doing it a little different from how we did it in class, but I think Lawson & Michelsohn make some interesting comments. A **spin structure** on an oriented Riemannian n-manifold X is a pair $(P_{\mathrm{Spin}}(X), \xi)$ where:

This actually only covers the case of $n > 2$. If $n = 2$, then we use SO(2) instead of Spin(n), and if $n = 1$ then $P_{\mathrm{SO}}(X) \cong X$ and a spin structure is any 2-fold covering.

(i) $P_{\mathrm{Spin}}(X)$ is a principal Spin(n)-bundle.

(ii) $\xi\colon P_{\mathrm{Spin}}(X) \to P_{\mathrm{SO}}(X)$ is a 2-sheeted covering, where $P_{\mathrm{SO}}(X)$ is the principal SO(n)-bundle determined by the orientation of X from the orthonormal frame bundle associated to X.

(iii) For every $p \in P_{\mathrm{Spin}}(TX)$ and $g \in \mathrm{Spin}(n)$, we have

$$\xi(pg) = \xi(p)\xi_0(g),$$

where $\xi_0\colon \mathrm{Spin}(n) \to \mathrm{SO}(n)$ is the double cover.

The rough idea behind this is that this is equivalent to the second Stiefel-Whitney class of TX being trivial, and tells us that the structure group is 1-connected. Contrast this in the case where the first Stiefel-Whitney class of TX is trivial is equivalent to TX being orientable, which means that the structure group is 0-connected! Thus in short, we see:

Observation 3. *A spin manifold is a Riemannian manifold X with trivial first and second Stiefel-Whitney classes.*

A (real) **spinor bundle** for a spin n-manifold X is a triple $(S(X), M, \mu)$ where:

(i) M is a left-module for $\mathrm{C}\ell(\mathbb{R}^n)$.

(ii) $\mu\colon \mathrm{Spin}(n) \to \mathrm{SO}(M)$ is the representation given by left multiplication by elements of $\mathrm{Spin}(n) \subset \mathrm{C}\ell^0(\mathbb{R}^n)$.

(iii) $S(TX)$ is the bundle

$$S(X) \overset{\text{def}}{=} \frac{P_{\mathrm{Spin}}(X) \times M}{\mathrm{Spin}(n)},$$

where the group action of $g \in \mathrm{Spin}(n)$ on $(p, m) \in P_{\mathrm{Spin}}(X) \times M$ is given by:

$$g \curvearrowright (p, m) \overset{\text{def}}{=} (pg^{-1}, \mu(g)m).$$

A **complex spinor bundle** is defined analogously with $\mathrm{C}\ell(\mathbb{R}^n) \otimes \mathbb{C}$ in place of $\mathrm{C}\ell(\mathbb{R}^n)$. It turns out that a connection on $P_{\mathrm{SO}}(X)$ may be lifted to connection on $P_{\mathrm{Spin}}(X)$, and thus any spinor bundle S inherits a connection ∇^s as well. After some work we arrive at a notion for curvature on spinor bundles:

Observation 4 ([2] Th.4.15, pg. 110)**.** *If Ω is the curvature 2-form on $P_{\mathrm{SO}}(X)$ and $S(X)$ a spinor bundle on X, then the curvature R^s of $S(X)$ is given locally on a section $\sigma \in S(E)$ and vectors $v, w \in T_x X$ by:*

$$R^s_{v,w}(\sigma) = \frac{1}{2}\sum_{i<j}\langle R_{v,w}(e_i), e_j\rangle e_i e_j \sigma,$$

where $(e_1, \ldots, e_n)$ is a local section of $P_{\mathrm{SO}}(X)$.

Given a spin manifold and a spinor bundle $S(X)$, we have an associated **Dirac operator** which is $\partial\!\!\!/ \colon \Gamma(S(X)) \to \Gamma(S(X))$ given by:

$$\partial\!\!\!/(\sigma) \overset{\text{def}}{=} \sum_{j=1}^{n} e_j \cdot \nabla^s_{e_j}\sigma,$$

the dot indicating the Clifford multiplication. If X is a compact spin manifold of dimension $4k$ with the complex spinor bundle $S_{\mathbb{C}}(X)$ and associated (complex) Dirac operator $\partial\!\!\!/$, we can split the spinor bundle into **chiral spinor bundles**,

$$S_{\mathbb{C}}(X) \cong S^+_{\mathbb{C}}(X) \oplus S^-_{\mathbb{C}}(X),$$

so that $\partial\!\!\!/ = \partial\!\!\!/^+ \oplus \partial\!\!\!/^-$. The following can be shown using the **Atiyah-Singer Index Theorem**:

Observation 5 ([2] Ex.6.3, pg.137)**.** *The operator $\partial\!\!\!/^+$ is an elliptic differential operator, whose index is an integer called the* ***$\hat{A}$-genus****,*

$$\text{index}(\partial\!\!\!/^+) = \hat{A}(X).$$

In general, the $\hat{A}$-genus is not an integer, but for compact spin manifolds, it is! Here, this index works only for dimensions $4k$, though!

To add a *twist* to this idea, we can take any bundle $E \to X$ and now look at $S^{\pm}_{\mathbb{C}}(X) \otimes E$. In particular now, the associated Dirac operator is the **twisted Dirac operator**, $\partial\!\!\!/^+_E \colon \Gamma(S^+_{\mathbb{C}}(X) \otimes E) \to \Gamma(S^-_{\mathbb{C}}(X) \otimes E)$, whose associated index is

$$\text{index}(\partial\!\!\!/^+_E) = \int_X \operatorname{ch} E \cdot \hat{A}(TX),$$

where $\hat{A}(TM)$ is the (total) **$\hat{A}$-class** of TX. If you do not know or forget what the $\hat{A}$-class is, continue to the next subsection.

2.2 Interlude: $\hat{A}$

Given a vector bundle $E \to X$, we can define the characteristic class $\hat{A}(E) \in H^*(X; \mathbb{Q})$ via the properties:

(i) $\hat{A}(f^*E) = f^*(\hat{A}(E))$, for any map $f\colon \tilde{X} \to X$.

(ii) $\hat{A}(E \oplus F) = \hat{A}(E) \cdot \hat{A}(F)$.

(iii) Given a complex line bundle $\ell \to X$ with Euler class $x \in H^2(X;\mathbb{Q})$, we have:

$$\hat{A}(\ell) = \frac{x/2}{\sinh(x/2)}.$$

For any oriented $4k$-manifold X, we define the **$\hat{A}$-genus** of X to be

$$\hat{A}(X) \overset{\text{def}}{=} \int_X \hat{A}(TX).$$

2.3 Introducing curvature into the equation

In this subsection we introduce the machinery that will be doing the heavy lifting for us later. Namely, we want to somehow introduce the notion of curvature into this set up of ours.

There is another notion of *Laplacian* on $\mathrm{C}\ell(X)$ besides the Dirac Laplacian $\not{\partial}^2$, discovered by Salomon Bochner. For any Riemannian vector bundle $E \to X$, define the **second covariant derivative** ∇^2 via:

$$\nabla^2_{v,w}\sigma \overset{\text{def}}{=} \nabla_v \nabla_w \sigma - \nabla_{\nabla_v w}\sigma \in \Gamma(E),$$

for vector fields v and w on X, and a section $\sigma \in \Gamma(E)$. Fixing σ, $\nabla^2_{\cdot,\cdot}\sigma$ is a section of $T^*X \otimes T^*X \otimes E$ and we can take the trace of it, which leads us to this alternative Laplacian called the **connection Laplacian**:

$$\nabla^*\nabla\sigma \overset{\text{def}}{=} -\operatorname{tr}(\nabla^2_{\cdot,\cdot}\sigma).$$

Observation 6. *The connection Laplacian $\nabla^*\nabla$ is elliptic with symbol $\sigma(\nabla^*\nabla)\xi = \|\xi\|^2$.*

Now the connection to curvature is as follows. If we consider the difference $\not{\partial}^2 - \nabla^*\nabla$ between our two notions of Laplacian, then we get a zero-order operator which can be expressed in terms of curvature.

Fun fact: if you use the Hodge Laplacian and the connection Laplacian on the tangent bundle, TX, then we have the identity:

$$\Delta - \nabla^*\nabla - \mathrm{Ric},$$

for the Ricci curvature.

Theorem 2.1 ([2] Th.8.2, pg.155)**.** *Given an orthonormal tangent frame (e_i), we have*

$$\not{\partial}^2 - \nabla^*\nabla = \frac{1}{2}\sum_{j,k=1}^{n} e_j \cdot e_k \cdot R^s_{e_j,e_k}.$$

This is the **Bochner identity**.

This is all in a more general setting than we really need, so we can now be a bit more specific. Consider a compact spin manifold X with Spinor bundle $S(X)$. Recall that the scalar curvature of X is given by the trace of the Ricci curvature. After a few mundane computations which exploit symmetries of $R_{uv}w$:

Observation 7 ([2] Th.8.8, pg.160; Th.8.17, pg.164)**.** *In the special case of a spin manifold, the Bochner identity yields the* **Lichnerowicz formula***:*

$$\partial\!\!\!/^2 - \nabla^*\nabla = \frac{1}{4}Scal.$$

In the twisted case, we get:

$$\partial\!\!\!/_E^2 - \nabla^*\nabla = \frac{1}{4}Scal + \mathfrak{R}^E,$$

where $\mathfrak{R}^E\colon S(X)\otimes E \to S(X)\otimes E$ *is defined by the formula:*

$$\mathfrak{R}^E(\sigma\otimes\epsilon) \stackrel{\text{def}}{=} \frac{1}{2}\sum_{j,k=1}^{n}(e_je_k\sigma)\otimes(R^E_{e_j,e_k}\varepsilon).$$

Using the Lichnerowicz formula, we can already get a restriction based on scalar curvature.

Theorem 2.2 ([2] Th.8.11, pg.161)**.** *Let X be a compact spin manifold of dimension $4k$. If X admits a metric of positive scalar curvature, then $\hat{A}(X) = 0$.*

3 The Geroch conjecture

The handy part of the Lichnerowicz will be that assuming positive scalar curvature will allow us to force (via an estimation) a certain twisted Dirac operator to have zero kernel which will then in turn force its index to be zero. How we then use this to prove the Geroch conjecture is to derive a contradiction by computing the index of the aforementioned twisted Dirac operator in a different way to yield that it is non-zero.

3.1 Enlargeable manifolds

We will actually prove the analogue of the Geroch conjecture for a larger class of manifolds called "enlargeable manifolds". The idea behind this class is that an enlargeable manifold admits covering spaces which are "large in all directions".

In particular, Gromov had the idea of looking at spaces which admitted for every $L > 0$ a covering space C_L with a continuous map $f\colon [0,1]^n \to C_L$ such that the distance between any two opposite faces of the cube- $[0,1]^n$ is at least L. Instead of using this particular formulation, Gromov and Lawson looked at a dual concept of ε-contracting maps.

We say that a map $f\colon X \to Y$ is **ε-contracting** if for all $v \in TX$ we have:

$$\|f_* v\| \leqq \varepsilon \|v\|.$$

Because constant maps are ready examples, we want to look at ones of non-zero degree. In particular, given a ε-contracting map $f\colon X \to Y$ of non-zero degree between compact spaces, we intuitively understand X to be "bigger" than Y.

With this in mind, we define a compact Riemannian n-manifold to be **enlargeable** if for all $\varepsilon > 0$ there is an orientable Riemannian covering space with a ε-contracting map onto S^n (with constant curvature 1) which is constant outside of a compact set, and of non-zero degree. In particular, if we can always find the covering space to be compact, then we say it is **compactly enlargeable**.

Observation 8. *The n-torus (viewed as the flat, square torus) is not only enlargeable (considering its universal cover), but is also compactly enlargeable as $C_k \overset{\text{def}}{=} \mathbb{R}^n/(k\mathbb{Z})^n$ is a finite k^n-fold covering space for the n-torus, and define a ε-contracting map $f\colon C_k \to S^n$ by mapping the centre of the inscribed ball to the north pole of S^n, and the complement of said ball collapsed to the south pole (in particular, $\varepsilon = \pi/k$).*

It is a fact of enlargeable manifolds that enlargeability is independent of the metric and depends only on the homotopy type of the manifold. Moreover, enlargeability is closed under products and connected sums, and any manifold admitting a map of non-zero degree onto an enlargeable manifold is also enlargeable ([2] Th.5.3, pg.303).

The theorem of interest that we will sketch a proof for is the following:

Theorem 3.1 (Gromov-Lawson)**.** *A compactly enlargeable spin manifold X cannot carry a metric of positive scalar curvature.*

However, we do not need to worry about X being spin, in particular, just that the covering space we use in the proof (following section) is spin.

3.2 The proof

We will now give a short sketch of the proof of Theorem 3.1 as it appears in Lawson and Michelsohn ([2] Th.5.5, pg.306, where all the dirty details can be found). The proof is via contradiction and revolves around calculating the index of a Dirac operator in two different ways.

Proof. Assume that such an X has a metric with $Scal \geqq k_0$, for some $k_0 > 0$. First we note that without loss of generality, X has even dimension $2n$. If it didn't, we could consider $X \times S^1$ which does. Take the following observation for granted:

> **Observation 9.** *It is possible to find a complex vector bundle E_0 over S^{2n} with the property that the top Chern class $c_n(E_0) \neq 0$. In particular,*
>
> $$\operatorname{ch} E_0 = \dim E_0 + \frac{1}{(n-1)!} c_n(E_0),$$
>
> *because S^n has trivial cohomology in for degrees $1 \leqq i < n$.*

We can give it a unitary connection ∇^{E_0} and denote its corresponding curvature by R^{E_0}.

Because X is compactly enlargeable, let $\varepsilon > 0$ and choose a finite orientable covering $\tilde{X} \to X$ with a corresponding ε-contracting map $f\colon \tilde{X} \to S^{2n}$ of non-zero degree. We pull back the bundle E_0 to $E \overset{\text{def}}{=} f^* E_0$, which has a corresponding connection $\nabla^E \overset{\text{def}}{=} f^* \nabla^{E_0}$.

Take a (complex) spinor bundle $S_{\mathbb{C}}(\tilde{X}) \cong S_{\mathbb{C}}^+ \oplus S_{\mathbb{C}}^-$ and consider the twisted spinor bundle $S_{\mathbb{C}}(\tilde{X}) \otimes E$ with Atiyah-Singer operator $\partial\!\!\!/_E$. Recall the Lichnerowicz formula in this case as:

$$\partial\!\!\!/_E^2 - \nabla^* \nabla = \frac{1}{4} Scal + \mathfrak{R}^E. \qquad (\dagger)$$

Another observation can be made:

> **Observation 10.** *$\mathfrak{R}^E$ depends linearly on the components of the curvature tensor R^E of E and is a symmetric bundle endomorphism of $S_{\mathbb{C}}(\tilde{X}) \otimes E$.*

There is a constant $k_n = k(n)$ depending on n such that:

$$\|\mathfrak{R}^E\| \overset{\text{def}}{=} \sup\{g(\mathfrak{R}^E \sigma, \sigma) \mid \|\sigma\| = 1\} \leqq k_n \|R^E\|,$$

but R^E is the pull-back of R^{E_0} by the ε-contracting map f. Thus, with some work we observe:

> **Observation 11.**
>
> $$\|\mathfrak{R}^E\| \leqq k_n \|R^E\| \leqq k_n \varepsilon^2 \|R^{E_0}\|.$$

Because the sphere S^{2n} is of constant curvature 1 and we can pick $\varepsilon < \sqrt{k_0/2k_n}$, we see that:

$$\|\mathfrak{R}^E\| < \frac{1}{4}k_0.$$

But then by (†), we have that if $\sigma \neq 0$, then $\not{\partial}_E(\sigma) \neq 0$. Thus $\not{\partial}_E^+$ and $\not{\partial}_E^-$ have trivial kernel, so:

$$\operatorname{index} \not{\partial}_E^+ = \dim\ker \not{\partial}_E^+ - \dim\ker \not{\partial}_E^- = 0.$$

Note that we have basically just given a proof for Theorem 2.2. Now we compute the index topologically using the Atiyah-Singer index theorem! We compute it as follows:

$$\begin{aligned}
\operatorname{index}(\not{\partial}_E^+) &= \int_{\tilde{X}} \operatorname{ch} E \cdot \hat{A}(T\tilde{X}) \\
&= \int_{\tilde{X}} \left(\dim E + \frac{1}{(n-1)!}c_n(E)\right) \cdot \hat{A}(T\tilde{X}) \\
&= \int_{\tilde{X}} \dim E \cdot \hat{A}(\tilde{X}) + \frac{1}{(n-1)!}c_n(E) \\
&= \frac{1}{(n-1)!}\int_{\tilde{X}} c_n(E), \quad \text{(applying something like Theorem 2.2)} \\
&= \frac{1}{(n-1)!}\int_{\tilde{X}} c_n(f^*E_0) \\
&= \frac{1}{(n-1)!}\int_{\tilde{X}} f^*(c_n(E_0)) \\
&= \frac{\deg f}{(n-1)!}\int_{S^{2n}} c_n(E_0),
\end{aligned}$$

which is non-zero, recalling that $\deg f$ is in particular non-zero as well! Thus we note a contradiction and there is no metric with $Scal > 0$! ■

Note that Gromov & Lawson have a more general result for (not necessarily compactly) enlargeable spin manifolds, but it uses the relative index theorem and so in Lawson and Michelsohn's book, they focus on the case above. As well, Gromov-Lawson also shows that any metric with $Scal \geqq 0$ on X must be flat.

References

[1] Larry Guth. Metaphors in systolic geometry. In *Proceedings of the International Congress of Mathematicians 2010 (ICM 2010) (In 4 Volumes) Vol. I: Plenary Lectures and Ceremonies Vols. II–IV: Invited Lectures*, pages 745–768. World Scientific, 2010.

[2] H Blaine Lawson and Marie-Louise Michelsohn. *Spin geometry*, volume 38. Princeton university press, 2016.