Gel’fand and his algebras

Luke Volk

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This document is an amateur’s attempt to fill in a modern treatment of topic of Banach algebras and the Gel’fand transform with historical details. Historical remarks will be indicated by *a font like this* for the sake of interest.

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1 Where we pick up the story

1.1 Normed vector spaces

Early in the history of functional analysis, special spaces like ℓ^p, L^p, or $C[a, b]$ dominated the interest of mathematicians. It wasn't until the late 1910s that mathematicians decided to take a step towards generality.

One of the first, Eduard Helly (1884–1943) considered in 1921 more general "normed sequence spaces", contrasting with the work of Schmidt and Riesz. Building off of Helly's work, it was more or less natural to move the spotlight to arbitrary normed vector spaces. Both Hans Hahn (1879–1934) and Stefan Banach (1892–1945) did just that, independent of each other, in particular restricting themselves to normed vector spaces which are complete with respect to the norm.

Helly considered vector subspaces of $\mathbb{C}^{\mathbb{N}}$, though he did not use the term "norm" nor even the modern notation for a norm. Helly was also the first to give an example of non-reflexive Banach spaces.

1.2 Spectral theory

Our pseudo-historical approach to the introduction of algebras to analysis will follow closely the history of spectral theory. It was more or less David Hilbert (1862–1943) who was responsible for spurring on spectral theory in his work in 1906. He dealt with the subject through bilinear forms, but Erik Ivar Fredholm (1866–1927) opted for operators. Frigyes Riesz (1880–1956) took it upon himself in 1913 to translate much of Hilbert's work on spectral theory into the language of operators.

Though Riesz did not end up using uniform convergence from the operator norm, but rather strong convergence.

It was 1926 when (a very young) John von Neumann (1903–1957) arrived at Göttingen as Hilbert's assistant. In his historical account of functional analysis, expert storyteller Dieudonné weaves the tale:

> These were the hectic years during which quantum mechanics was developing at breakneck speed, with a new idea popping up every few weeks from all over the horizon. The theoretical physicists who were developing the new theory were groping for adequate mathematical tools, trying in sucession infinite matrices without any consideration of convergence (as late as 1924, most physicists did not even know what a *finite* matrix was!), differential operators, "continuous" matrices (whatever that might mean) etc. It finally dawned upon them that their "observables" had properties which made them look like Hermitian operators in Hilbert space, and that, by extraordinary coincidence, the "spectrum" of Hilbert (a name which he had apparently chosen from a superficial analogy) was to be the central conception in the explanation of the "spectra" of atoms.

And thus von Neumann was sucked into helping Hilbert to these physical ends. It was von Neumann who had first generalized the concept of a Hilbert space away from matrices towards an axiomatic definition. This paved the way to the modern development of the spectral theory of normal and Hermitian operators on Hilbert spaces on which he subsequently published several papers on between 1929 and 1932. One especially key habit von Neumann had was to work intrinsically from axioms rather than from examples as the pioneers before him did.

Dieudonné notes that this was useful to the end of quantum mechanics since most operators there were defined on a proper subspace.

2 Algebras

2.1 Normed and involutive algebras

Let H be a Hilbert space. Riesz was likely the first to consider the *algebra*

$$L(H) = \{T\colon H \to H \mid T \text{ linear and continuous}\},$$

the algebra of endomorphisms on H. By **algebra**, we mean a $\mathbb{C}$-vector space with an associated bilinear product operation between vectors, and in particular $L(H)$ has a product given by composition. *It is probably more accurate to say that Riesz was the first who considered $L(H)$ as an algebra in addition to using its norm and strong topology, using it in the context of spectral theory and normal operators.*

It is worth mentioning that we are explicitly not including unital in our definition. An algebra lacking a unit is not exactly difficult to remedy, as we can always embed a non-unital algebra $\mathcal{A}$ into a unital one, $\mathcal{A} \oplus \mathbb{C}\mathbb{1}$, where $\mathbb{1}$ will be our unit.

Observation 1. *Any non-unital algebra can be embedded into a unital algebra.*

Associated to $L(H)$ is the operator norm,

$$\|T\| := \inf\{M > 0 \mid \forall x \in H \,.\, \|Tx\|_H \leqq M\|x\|_H\}, \quad T \in L(H),$$

which behaves nicely with regards to the product of the algebra:

$$\|ST\| \leqq \|S\|\|T\|, \quad S, T \in L(H).$$

An *associative* algebra with such a norm (i.e. sub-multiplicative) is called a **normed algebra**. With regards to the unital extension $\mathcal{A} \oplus \mathbb{C}\mathbb{1}$ from before, any unital normed algebra can be given an equivalent norm such that $\|\mathbb{1}\| = 1$.

Observation 2. *Any unital normed algebra can be given an equivalent norm such that $\|\mathbb{1}\| = 1$.*

Example 2.1. Already given is the normed algebra $L(H)$ of endomorphisms of a Hilbert space H. Another reoccurring example will be the space $B(X)$ of bounded complex-valued functions on a non-empty set X with the norm

$$\|f\| := \sup_{x \in X} |f(x)|,$$

which has the point-wise product:

$$(fg)(x) := f(x)g(x), \quad f, g \in B(X).$$

The normed algebra $L(H)$ was also an interest of von Neumann's who was interested in particular about its "subalgebras". A **subalgebra** of an algebra is a subset which is closed under the algebra product. Besides its product, $L(H)$ also has the adjoint operation,

$$T \mapsto T^*, \quad T^* \text{ defined by } \forall x, y \in H \,.\, \langle Tx, y\rangle = \langle x, T^* y\rangle.$$

The adjoint has the following properties:

(i) The adjoint operation is conjugate-linear.

(ii) For $T, S \in L(H)$ we have $(ST)^* = T^* S^*$.

(iii) $(T^*)^* = T$.

With this additional structure we can then ask about subalgebras closed under this operation, called **involutive subalgebras**, or ***-subalgebras**, *and these were indeed the types of subalgebras von Neumann occupied himself with—mind you, 5 years before the elementary theory of normed algebras was developed!*

Dieudonné notes that Artin and Noether had been working on rings with descending chain conditions at the time, applying the theory to linear representations of groups and number theory. This was the inspiration for von Neumann who wanted to see if something similar was possible with involutive subalgebras of $L(H)$, substituting the chain condition with suitable topological restrictions—this was the debut of von Neumann algebras.

2.2 Banach algebras and the spectrum

The development of the theory of normed algebras we owe to Israel Gel'fand (1913–2009) in 1941. Gel'fand's key idea was to extend spectral theory to the elements of a unital normed algebra $\mathcal{A}$ by merely implementing Riesz's definition of the **spectrum**:

$$\sigma(x) := \{\zeta \in \mathbb{C} \mid x - \zeta \mathbb{1} \text{ is not invertible in } \mathcal{A}\}, \quad x \in \mathcal{A},$$

where $x \in \mathcal{A}$ is invertible if there exists an $y \in \mathcal{A}$ such that $xy = yx = \mathbb{1}$ (it is necessarily unique).

Key to Gel'fand's results was the completeness of the normed algebra that he used. A normed algebra which is complete with respect to the topology induced by its norm is called a **Banach algebra**.

Example 2.2. Both $L(H)$ and $B(X)$ are Banach algebras, and we have the following new examples:

More generally, $L(X)$ is Banach if and only if X is.

(1) If $\mathcal{A}$ is a Banach algebra, then a subalgebra $\mathcal{B} \subseteq \mathcal{A}$ is Banach if and only if $\mathcal{B}$ is closed in $\mathcal{A}$.

(2) The completion of a normed algebra is a Banach algebra via considering limits of Cauchy sequences.

(3) If X is locally compact and Hausdorff, then the set $C_c^0(X)$ of continuous functions on X with compact support is a normed subalgebra

of $B(X)$ whose closure is a Banach algebra $C(X)$ consisting of continuous functions on X which vanish at infinity.

If I am not mistaken (which I very well might be) the following fact (a proof of which can be found in e.g. Lang) is only a theorem of *Banach* algebras, rather than general normed algebras:

Observation 3. *If $\mathcal{A}$ is a unital Banach algebra, then*

$$\mathcal{A}^{\times} := \{x \in \mathcal{A} \mid x \text{ is invertible}\}$$

is an open subset of $\mathcal{A}$.

Using this, we have the following result about the spectrum for unital Banach algebras:

Proposition 2.3. *If $\mathcal{A}$ is a unital Banach algebra and $x \in \mathcal{A}$, then $\sigma(x)$ is a non-empty compact subset of $\mathbb{C}$.*

Sketch. For a contradiction, we suppose $\sigma(x) = \varnothing$. We define the resolvent function,

$$R(\zeta) = (x - \zeta)^{-1},$$

which we show is weakly holomorphic and bounded, so by Liouville's theorem it is constant. Hahn-Banach is then applied to show the contradiction that R is identically zero.

To show compactness, we show $\sigma(x)$ is closed and bounded.

Proof. Suppose for a contradiction that $\sigma(x) = \varnothing$. Consider the map $R\colon \mathbb{C} \to \mathcal{A}$ defined by:

$$R(\zeta) = (x - \zeta \mathbb{1})^{-1},$$

often called the **resolvent function** of x. It's easy to establish by direct computation that:

$$R(\zeta_1)R(\zeta_2) = \frac{R(\zeta_1) - R(\zeta_2)}{(\zeta_1 - \zeta_2)}.$$

If $\phi \in \mathcal{A}^*$, then for $\zeta_0 \in \mathbb{C}$ we have:

$$\lim_{\zeta \to \zeta_0} \frac{\phi(R((\zeta)) - \phi(R(\zeta_0))}{\zeta - \zeta_0} = \lim_{\zeta \to \zeta_0} \phi(R(\zeta)R(\zeta_0)) = \phi(R(\zeta_0)^2),$$

thus $\phi \circ R$ is a holomorphic function. Note that for $|\zeta| > \|x\|$, we have:

$$R(\zeta) = -\frac{1}{\zeta}(\mathbb{1} - \zeta^{-1}x)^{-1} = -\frac{1}{\zeta}\sum_{n=0}^{\infty} \frac{x^n}{\zeta^n} \underset{|\zeta| \to \infty}{\longrightarrow} 0.$$

We then apply Liouville's theorem:

Every bounded entire function must be constant.

In particular, $\phi \circ R \equiv 0$ for all $\phi \in \mathcal{A}^*$. By the Hahn-Banach theorem, for each $\zeta \in \mathbb{C}$ there exists $\phi \in \mathcal{A}^*$ such that $\phi(R(\zeta)) = \|R(\zeta)\|$ and $\|\phi\| = 1$, so $R(\zeta) = 0$ for all $\zeta \in \mathbb{C}$, so $R \equiv 0$, a contradiction! Thus $\sigma(x) \neq \varnothing$.

To prove $\sigma(x)$ is compact, note that for $|\zeta| > \|x\|$ we have:

$$x - \zeta\mathbb{1} = -\zeta(\mathbb{1} - \zeta^{-1}x),$$

where $(\mathbb{1} - \zeta^{-1}x)^{-1} = \sum_{n=0}^{\infty} \frac{x^n}{\zeta^n}$, so $x - \zeta\mathbb{1} \in \mathbb{C} \setminus \sigma(x)$ and

$$\sigma(x) \subseteq \{\zeta \in \mathbb{C} \mid |\zeta| \leqq \|x\|\}$$

is clearly bounded. To show $\sigma(x)$ is closed (and thus compact), consider the function $S\colon \zeta \mapsto x - \zeta\mathbb{1}$ so that we write $\zeta \in \mathbb{C} \setminus \sigma(x) \iff \zeta \in S^{-1}(\mathcal{A}^\times)$. But S is provably continuous and so by the fact that $\mathcal{A}^\times$ is open, $\mathbb{C} \setminus \sigma(x)$ is as well, so $\sigma(x)$ is closed. Hence $\sigma(x)$ is compact. ■

Theorem 2.4 (Polynomial spectral mapping theorem)**.** *Suppose that* $p \in \mathbb{C}[z]$. *Then:*

$$\sigma(p(x)) = p(\sigma(x)) = \{p(\zeta) \mid \zeta \in \sigma(x)\}.$$

Proof. Let $\deg p \geqq 1$ and $\alpha, \lambda, \lambda_1, \ldots, \lambda_n \in \mathbb{C}$ such that:

$$p - \lambda = \alpha(z - \lambda_1)\ldots(z - \lambda_n).$$

Thus $p(x) - \lambda\mathbb{1} = \alpha(x - \lambda_1\mathbb{1})\ldots(x - \lambda_n\mathbb{1})$. If $x_1, \ldots, x_n \in \mathcal{A}$ mutually commute, then it's a straightforward exercise in symbol pushing that:

$$a_1 a_2 \ldots a_n \in \mathcal{A}^\times \iff \alpha_i \in \mathcal{A}^\times, \quad 1 \leqq i \leqq n.$$

If $p(x) - \lambda\mathbb{1}$ is non-invertible ($\lambda \in \sigma(p(x))$), then we can find some $1 \leqq i \leqq n$ so that $x - \lambda_i\mathbb{1}$ is also non-invertible ($\lambda_i \in \sigma(x)$), where we see:

$$p(\lambda_i) = \lambda \implies \lambda \in p(\sigma(x)).$$

Conversely, if $\mu \in \sigma(x)$ and $\lambda = p(\mu)$, then it follows from above that $\mu = \lambda_i$ for some $1 \leqq i \leqq m$, so $p(x) = \lambda\mathbb{1}$ is non-invertible. Hence $p(\mu) \in \sigma(p(x))$. ■

Now given that the spectrum $\sigma(x)$ of an $x \in \mathcal{A}$ is non-empty, we define the **spectral radius** of x by:

$$\rho(x) := \sup_{\zeta \in \sigma(x)} |\zeta|.$$

Gel'fand had obtained a remarkable formula for the spectral radius:

Theorem 2.5 (Spectral radius formula)**.** *Suppose that $\mathcal{A}$ is a unital Banach algebra and $x \in \mathcal{A}$. Then:*

$$\rho(x) = \lim_{n\to\infty} \|x^n\|^{1/n}.$$

Sketch. For one inequality, we again appeal to the fact that the resolvent function is weakly holomorphic on the complement of $D(0; \rho(x))$. The uniform boundedness principle allows us to conclude that the weakly bounded set

$$\{x^n/\zeta^n \mid n \in \mathbb{N}\}$$

is sup-bounded. The other inequality is given by the spectral mapping theorem.

Proof. Similar to what we had shown in the proof of Proposition 2.3, we can show that the resolvent function $R\colon \mathbb{C} \setminus \sigma(x) \to \mathcal{A}^\times$ of x is weakly holomorphic on the complement of the disc $D(0; \rho(x))$. If $\phi \in \mathcal{A}^*$, then in the complement of the disc $D(0; \|x\|)$ we have the series expansion:

$$\phi(R(\zeta)) = -\sum_{n=0}^{\infty} \frac{\phi(x^n)}{\zeta^{n+1}},$$

which vanishes at infinity. Thus the series expansion extends to the entirety of the complement of $D(0; \rho(x))$. Thus

$$\lim_{n\to\infty} \frac{\phi(x^n)}{\zeta^n} = 0.$$

Thus $\{x^n/\zeta^n \mid n \in \mathbb{N}\}$ is a weakly bounded set. Hence by the uniform boundedness principle (applied to the family $\{\phi \in \mathcal{A}^* \mapsto \phi(x^n/\zeta^n) \mid n \in \mathbb{N}\}$),

$$\sup_{n\in\mathbb{N}} \frac{\|x^n\|}{|\lambda|^n} = M < \infty.$$

Hence we have:

$$\|x^n\|^{1/n} \leqq K^{1/n}|\zeta| \underset{|\lambda|\to\rho(x)}{\Longrightarrow} \limsup_{n\in\mathbb{N}} \|x^n\|^{1/n} \leqq \rho(x).$$

Conversely, from the spectral mapping theorem and $\rho(x) \leqq \|x\|$ we have:

$$\rho(x) = \rho(x^n)^{1/n} \leqq \|x^n\|^{1/n} \implies \rho(x) \leqq \liminf_{n\in\mathbb{N}} \|x^n\|^{1/n}.$$

Hence we have:

$$\limsup_{n\to\infty} \|x^n\|^{1/n} \leqq \rho(x) \leqq \inf_{n\in\mathbb{N}} \|x^n\|^{1/n},$$

and the limit exists. ∎

3 Gel'fand theory

3.1 Characters and maximal ideals

An algebra is said to be **commutative** if the product operation of the algebra is commutative. *Likely inspired by the theory of Abelian groups, Gel'fand turned to studying commutative Banach algebras via defining characters for a commutative Banach algebra.* If $\mathcal{A}$ is a commutative Banach algebra, then a **character** of $\mathcal{A}$ is a non-zero map $\chi\colon \mathcal{A} \to \mathbb{C}$ satisfying:

(i) χ is linear.

(ii) For all $x, y \in \mathcal{A}$, $\chi(xy) = \chi(x)\chi(y)$.

More generally, a map $\chi\colon \mathcal{A} \to \mathcal{B}$ of algebras satisfying the above is called an **algebra homomorphism**. Thus characters are simply non-zero complex-valued homomorphisms on $\mathcal{A}$.

If $\mathcal{A}$ is unital, then for any character χ and $x \in \mathcal{A}$:

$$\chi(x) = \chi(x\mathbb{1}) = \chi(x)\chi(\mathbb{1}),$$

and thus $\chi(\mathbb{1}) = 1$. As a result,

$$x - \chi(x)\mathbb{1} \in \ker\chi \implies \chi(x) \in \sigma(x),$$

for each $x \in \mathcal{A}$. In particular, this means that $|\chi(x)| \leqq \|x\|$, so the character χ is necessarily continuous. This same argument extends to a non-unital Banach algebra by looking at the unitization $\mathcal{A} \oplus \mathbb{C}\mathbb{1}$ and the character $\tilde{\chi}(x, \zeta) := \chi(x) + \zeta$.

Observation 4. *A character χ of a Banach algebra is continuous with $\|\chi\| = 1$, and $\chi(x) \in \sigma(x)$.*

As per usual, the kernel of a homomorphism is an important object to study. In the context of an algebra homomorphism $\phi\colon \mathcal{A} \to \mathcal{B}$, the kernel $I = \ker\phi$ satisfies the conditions:

(i) I is a linear subspace of $\mathcal{A}$.

(ii) If $x \in I$ and $y \in \mathcal{A}$, then $xy, yx \in I$.

More generally, a subset I satisfying these conditions is known as an (two-sided) **ideal** of $\mathcal{A}$. Thus $\ker\phi$ is an ideal of $\mathcal{A}$.

Suppose that χ is a character of a unital Banach algebra $\mathcal{A}$. The kernel $I = \ker\chi$ is an ideal and is necessarily proper (otherwise $\chi \equiv 0$). If $y \notin I$, then for any $x \in I$, we can write x as:

$$x = \underbrace{\left(x - y\frac{\chi(x)}{\chi(y)}\right)}_{\in I} + \underbrace{y\frac{\chi(x)}{\chi(y)}}_{\in\mathbb{C}y},$$

which gives a decomposition $\mathcal{A} = I + \mathbb{C}y$. Thus if we take any ideal J containing both I and some element *not* in I, then $J = \mathcal{A}$. This means I is a **maximal** proper ideal. Thus any character corresponds to a maximal ideal.

Most often, people say I is maximal when they mean both maximal and proper because the "maximality" of $\mathcal{A}$ goes without saying.

Observation 5. *To each character of a unital Banach algebra, we can associate a maximal ideal.*

Clearly, if M is any maximal ideal of $\mathcal{A}$, then $M \cap A^\times = \varnothing$. Because $\mathcal{A} \setminus \mathcal{A}^\times$ is closed we have

$$M \subseteq \bar{M} \subsetneq \mathcal{A} \setminus \mathcal{A}^\times,$$

but M is maximal and so $\bar{M} = M$. Thus any maximal ideal is closed.

Observation 6. *The maximal ideals of a unital Banach algebra are closed subsets.*

The closed ideals of a Banach algebra $\mathcal{A}$ are very useful to the end of making new Banach algebras because then the quotient algebra $\mathcal{A}/I$ is also Banach:

Proposition 3.1. *If $\mathcal{A}$ is a Banach algebra with a closed ideal I, then $\mathcal{A}/I$ is a Banach algebra with respect to the norm:*

$$\|x + I\| = \inf_{a \in I} \|x + a\|.$$

Moreover, if $\mathcal{A}$ is unital and I is proper, $\mathcal{A}/I$ is also unital.

Proof. It is an easy exercise in algebra to show $\mathcal{A}/I$ is an algebra with the product $(x + I)(y + I) = (xy + I)$. Moreover, it is a *normed* algebra because:

$$\begin{aligned} \|xy + I\| &:= \inf_{a \in I} \|xy + a\| \\ &\leqq \inf_{a,b \in I} \|xy + \underbrace{xb + ay + ab}_{\in I}\| \\ &= \inf_{a,b \in I} \|(x + a)(y + b)\| \\ &\leqq \|x + I\|\|y + I\|, \end{aligned}$$

so $\mathcal{A}/I$ is a normed algebra. It is then a theorem of Banach spaces that if X is Banach and $Y \leq X$ is closed $\implies X/Y$ is Banach.

Finally, if I is proper, then it does not contain $\mathbb{1}$ and thus $\mathbb{1} + I$ operates as the unit. Moreover, a bonus is that $\|\mathbb{1}\| = 1 \implies \|\mathbb{1} + I\| = 1$. ■

In particular, since a maximal ideal $M \subset \mathcal{A}$ is closed, then $\mathcal{A}/M$ is a Banach algebra. Those who have done a little bit of ring theory might already see where this is going! Suppose that $\mathcal{A}$ is unital and commutative, M a maximal ideal, and $x + M \neq 0$ be a non-invertible element of $\mathcal{A}/M$. Appealing to commutativity, $x\mathcal{A} + M$ is a proper ideal of $\mathcal{A}$, and thus we have the inclusions:

$$M \subseteq x\mathcal{A} + M \subsetneq \mathcal{A},$$

and so by maximality of M, $x\mathcal{A} + M = M \implies x \in M$, so $x + M = 0$—a contradiction. Hence any non-zero element of $\mathcal{A}/M$ is invertible.

Observation 7. *If M is a maximal ideal of a commutative unital Banach algebra $\mathcal{A}$, then any non-zero element of $\mathcal{A}/M$ is invertible.*

Note that in this case, $\mathcal{A}/M$ cannot have a non-zero proper ideal. An algebra satisfying this property is called **simple**. In the commutative unital case, Gel'fand and Mazur showed that simple algebras are really—well, simple:

Theorem 3.2 (Gel'fand-Mazur)**.** *The only simple commutative unital Banach algebra (over $\mathbb{C}$) up to isomorphism is $\mathbb{C}$.*

Proof. Suppose that $x \in \mathcal{A}$ is not of the form $\zeta\mathbb{1}$ for $\zeta \in \mathbb{C}$. Let $\xi \in \sigma(x)$ (remember, it's non-empty!) and define an ideal

$$I = (x - \xi\mathbb{1})\mathcal{A},$$

which is proper in $\mathcal{A}$. Hence $I = \{0\}$ and

$$x - \xi\mathbb{1} = 0 \iff x = \xi\mathbb{1}.$$

Thus $\mathcal{A} \cong \mathbb{C}$. ■

Example 3.3. A quick application of the Gel'fand-Mazur theorem is that the quaternions are not a complex Banach algebra (though they *are* a real Banach algebra!). Indeed, the quaternions are a division algebra (each non-zero element is invertible) and hence it is simple (if an ideal contains an invertible element, then it is the entire algebra). Thus the quaternions would be a simple commutative unital Banach algebra, and thus be isomorphic to $\mathbb{C}$. This would be nonsensical.

Returning to the case where $\mathcal{A}$ is commutative unital and M is a maximal ideal, $\mathcal{A}/M$ is a simple commutative Banach algebra, and thus $\mathcal{A}/M = \mathbb{C}(\mathbb{1} + M)$. We can then consider the composition:

$$\mathcal{A} \xrightarrow{\pi} \mathcal{A}/M \xrightarrow{\varphi} \mathbb{C}$$

where π is the quotient map and $\varphi(\zeta \mathbb{1} + M) := \zeta$. The linear map $\chi = \varphi \circ \pi \colon \mathcal{A} \to \mathbb{C}$ then has the property:

$$\chi(xy) = \chi(x)\chi(y),$$

with $\ker \chi = M$. Thus χ is a character of $\mathcal{A}$.

Observation 8. *To each maximal ideal of a commutative unital Banach algebra, we can associate a character.*

Together with association of characters to maximal ideals (their kernels), this gives:

Theorem 3.4. *There is a bijective correspondence between the characters and maximal ideals of a commutative unital Banach algebra.*

It is extremely important that the commutativity assumption not be dropped. This can be seen with a relatively simple algebra:

Example 3.5. Consider the Banach algebra $M(n, \mathbb{C})$ of $n \times n$ complex matrices, $n > 1$. Let e_{ij} be the matrix which is zero everywhere except for 1 in the (i, j)-position. If χ is a character on $M(n, \mathbb{C})$, then for $i \neq j$:

$$0 = \chi(0) = \chi(e_{ij}^2) = \chi(e_{ij})^2,$$

and so $\chi(e_{ij}) = 0$. However, this means

$$\chi(e_{ii}) = \chi(e_{ij}e_{ji}) = \chi(e_{ij})\chi(e_{ji}) = 0,$$

and in particular,

$$\chi(\mathbb{1}) = \chi\left(\sum_{i=1}^{n} e_{ii}\right) = \sum_{i=1}^{n} \chi(e_{ii}) = 0,$$

a contradiction to χ being a character!

3.2 The Gel'fand transform

Dieudonné notes that Marshall Stone (1903–1989) had already considered the maximal ideals of Boolean rings in 1937. It was known that one could transfer the notion of spectrum to measures by way of projection-valued measures, and more generally to Boolean algebras.

A Boolean ring is a commutative ring R such that for each $x \in R$ we have $x^2 = x$ and $2x = 0$.

Stone noted in 1935 that Boolean algebras are in bijective correspondence with Boolean rings, and thus went on to consider how the notion of the spectrum passes to Boolean rings. In particular, the spectrum manifested itself as the set of maximal ideals in the Boolean ring, and he later introduced a topology on the spectrum by taking for each ideal the set of maximal ideals containing it, calling these the closed sets. In this respect, Gel'fand's approach to the spectrum of commutative unital Banach algebras was completely warranted at the time.

Suppose $\mathcal{A}$ is a commutative unital Banach algebra. Denote the set of characters on $\mathcal{A}$ by $\hat{\mathcal{A}}$. Note that $\hat{\mathcal{A}}$ is a subset of $\mathcal{A}^*$ and thus inherits a subspace topology (sometimes called the **Gel'fand topology**) from $\mathcal{A}^*$ given by the weak-* topology (which makes $\mathcal{A}^*$ into a Hausdorff space).

In particular, it makes it into a *topological group*.

Proposition 3.6. *If $\mathcal{A}$ is a commutative unital Banach algebra, then $\hat{\mathcal{A}}$ is a weak-* closed subset of the unit ball of A^*, and hence is weak-* compact.*

Proof. We have already shown that each $\chi \in \hat{A}$ has $\|\chi\| = 1$, and so $\hat{\mathcal{A}}$ is contained in the unit ball of $\mathcal{A}^*$. If $(\chi_\alpha) \subseteq \hat{\mathcal{A}} \to \chi \in A^*$, then for each $x \in \mathcal{A}$ we have:

$$\chi_\alpha(x) \to \chi(x).$$

Thus in particular for $x, y \in \mathcal{A}$ we have:

$$\chi(xy) = \lim_\alpha \chi_\alpha(xy) = \lim_\alpha \chi_\alpha(x)\chi_\alpha(y) = \chi(x)\chi(y),$$

so $\chi \in \hat{A}$. Hence $\hat{\mathcal{A}}$ is weak* closed. Thus by the Banach-Alaoglu theorem, $\hat{\mathcal{A}}$ is weak-* compact. ■

Recall that for any $\chi \in \hat{A}$ and $x \in \mathcal{A}$, we have:

$$\chi(x) \in \sigma(x).$$

We can write this in a different manner as follows. For each $x \in \mathcal{A}$, define a function $\hat{x} \colon \hat{A} \to \mathbb{C}$ by:

$$\hat{x}(\chi) := \chi(x), \quad \chi \in \hat{X}.$$

With this notation, the observation that $\chi(x) \in \sigma(x)$ for each $\chi \in \hat{\mathcal{A}}$ becomes:

$$\operatorname{im} \hat{x} \subseteq \sigma(x).$$

Now if $\zeta \in \sigma(x)$, suppose $x - \zeta\mathbb{1}$ is non-zero. Then define:

$$I = \{(x - \zeta\mathbb{1})y \mid y \in \mathcal{A}\},$$

which is a proper, non-zero ideal of $\mathcal{A}$. We can then apply Zorn's lemma to find a maximal ideal M of $\mathcal{A}$ containing I, and hence containing $x - \zeta\mathbb{1}$. This argument works in general for any non-invertible element of $\mathcal{A}$.

Observation 9. *An element of a commutative unital Banach algebra is invertible if and only if there exists a maximal ideal containing it.*

As a result of this, our correspondence between characters and maximal ideals of $\mathcal{A}$ yields a character χ with kernel $\ker \chi = M$, and hence:

$$\sigma(x) \subseteq \operatorname{im} \hat{x},$$

and so we actually have equality.

Observation 10. *If x is an element of a commutative unital Banach algebra, then:*

$$\sigma(x) = \operatorname{im} \hat{x}.$$

The functions $\hat{x}$ are all actually continuous (inherited from the weak-* topology), and thus are in $C(\hat{\mathcal{A}})$. Consider now the map $\Gamma\colon \mathcal{A} \to C(\hat{A})$ given by:

Recall that if X is locally compact and Hausdorff, then the set $C(X)$ is the closure of $C_c^0(X)$.

$$\Gamma(x) := \hat{x}.$$

Note for $x, y \in \mathcal{A}$ and $\zeta \in \mathbb{C}$ we have:

$$\widehat{\zeta x + y}(\chi) = \chi(\zeta x + y) = \zeta\chi(x) + \chi(y) = \zeta\hat{x}(\chi) + \hat{y}(\chi),$$

$$\widehat{xy}(\chi) = \chi(xy) = \chi(x)\chi(y) = \hat{x}(\chi)\hat{y}(\chi), \qquad \chi \in \hat{A},$$

and so Γ is an algebra homomorphism, which we call the **Gel'fand transform**. After noting that

$$\operatorname{im} \Gamma(x) = \sigma(x) \subset \{|\zeta| \leqq \|x\|\},$$

it is easy to see that:

$$\|\Gamma(x)\|_\infty = \rho(x) \leqq \|x\|.$$

Taking into account these properties, it's no wonder why the set $\hat{\mathcal{A}}$ is often called the **spectrum of $\mathcal{A}$**.

The Gel'fand transform is often called the **Gel'fand representation**. The algebra $C(X)$ of continuous complex-valued functions on X that vanish at infinity is in some sense an archetypal algebra. If X is compact, then $C(X)$ is exactly the algebra of all continuous complex-valued functions and it is unital. Because Γ is an algebra homomorphism, it in some sense is a representation (à la representations of groups and algebras) of $\mathcal{A}$, and thus one may study $\mathcal{A}$ by considering its image under the Gel'fand transform.

Problematically, at this point it is unclear whether Γ is injective or surjective. Failure of injectivity correlates with a loss of information across the Gel'fand transform, while failure of surjectivity means that we are working with a subalgebra of $C(\hat{A})$. It turns out that the Gel'fand transform is in general neither injective nor surjective.

That being said, a modest attempt at discussing the injectivity of Γ can be made. Note that $x \in \mathcal{A}$ is in every maximal ideal of $\mathcal{A}$ if and only if for all $\chi \in \hat{\mathcal{A}}$ we have $\chi(x) = 0$. Thus in this case, $\hat{x} = 0$, and so equivalently $x \in \ker \Gamma$. The intersection of all maximal ideals of $\mathcal{A}$ is called the **Jacobson radical** of $\mathcal{A}$, which we will notate as $\mathcal{J}(\mathcal{A})$. This presents the condition:

$$\Gamma \text{ is injective} \iff \mathcal{J}(\mathcal{A}) = \{0\}.$$

An algebra with a zero Jacobson radical is called **semi-primitive** and decomposes as a sub-direct product of "primitive rings", another object which gets a lot of attention from algebraists.

4 Odds and ends

In the sections that follow are things that we do not prove or explain in as much detail as one might want, but we do our best to try to try to get ideas across the best we can.

4.1 A cute example

Suppose that G is a group which we endow with a topology such that the multiplication (and inversion) are continuous (called a **topological group**), and such that it is locally compact and Hausdorff.

Consider the space $C_c^0(G)$ of continuous functions on G with compact support (with norm $\|\cdot\|_\infty$). For each $g \in G$ we have left-multiplication by g which defines a map $L_g : G \to G : h \mapsto gh$. Each L_g induces a map λ_g on $C_c^0(G)$ given by:

$$\lambda_g(f) = f \circ L_{g^{-1}}.$$

The use of the inverse is to force the relation $\lambda_{gh} = \lambda_g \circ \lambda_h$.

In a similar way that the Gel'fand transform defined a "representation" of $\mathcal{A}$ in $C(\hat{A})$, the association $g \mapsto \lambda_g$ is a group homomorphism and gives a "representation" of G in $C_c^0(G)$. A bounded linear functional f is said to be **left-translation invariant** if for all $g \in G$ we have:

$$f \circ \lambda_g = f.$$

The following theorem is far from obvious:

Theorem 4.1. *If G is a locally compact Hausdorff topological group, then there exists an $m \in C_c(G)^*$ which is left-translation invariant.*

Using the Riesz-Markov-Kakutani representation theorem:

> If X is a locally compact Hausdorff space, then for any positive linear functional ψ on $C_c^0(X)$, there is a unique regular Borel measure μ on X such that
>
> $$\psi(f) = \int_X f(x)\, d\mu(x), \quad f \in C_c^0(X).$$

we can find a unique measure μ on the Borel algebra of G such that:

$$m(f) = \int_G f(x)\, d\mu(x).$$

Because m was left-translation invariant, for each $g \in G$ we have that $\mu \circ L_g = \mu$ (that is, μ is also left-translation invariant). As it turns out, the pair (m, μ) are uniquely determined up to scaling by a positive scalar. The measure μ is known as the **left Haar measure** on G.

With the Haar measure as described above, we define $L^1(G, \mu)$ as $C_c^0(G)$ with the norm:

$$\|f\|_1 := \int_G |f(x)|\, d\mu(x),$$

which has a product given by convolution:

$$(f * g)(t) := \int_G f(x) g(s^{-1}t)\, d\mu(s).$$

Without proof, we have:

Proposition 4.2. *$L^1(G, \mu)$ with the product given by the above convolution is a Banach algebra.*

Now we suppose in particular that G is Abelian—this is key as it will allow us the commutativity condition for $L^1(G, \mu)$. Consider the set $\tilde{G}$ of non-zero group homomorphisms $\chi \colon G \to S^1$ (that is, **group characters**). To each $\chi \in \tilde{G}$ we can define:

$$\varphi_\chi(f) = \int_G f(x)\overline{\chi(x)}\, d\mu(x).$$

Without getting into the algebra, φ_χ is a character in $\widehat{L^1(G, \mu)}$ and indeed every character is of this form! Thus we have the bijective correspondence:

$$\begin{array}{ccc} \tilde{G} & \longleftrightarrow & \widehat{L^1(G, \mu)} \\ \chi & \longmapsto & \varphi_\chi \end{array}$$

The topology of $L^1(\tilde{G}, \mu)$ induces a topology on $\tilde{G}$, and $\tilde{G}$ itself is then a locally compact Hausdorff topological group, called the **Pontryagin dual** of G. Then we look at what we have just done: if we identify $\widehat{L^1(G, \mu)}$ with $\tilde{G}$, then the Gel'fand transform is identified with the map $\Gamma \colon L^1(G, \mu) \to C(\hat{G})$ given by:

$$\Gamma(f)(\chi) = \phi_\chi(f) = \int_G f(x)\overline{\chi(x)}\, d\mu(x).$$

Now suppose for the heck of it that we consider $G := \mathbb{R}$. It turns out that for each $y \in \mathbb{R}$, we have a character:

$$\chi_y(x) := e^{2\pi i x y}, \quad x \in \mathbb{R}.$$

The map $y \mapsto \chi_y$ is an homeomorphism as it turns out that every character is of this form. Thus we see:

$$\varphi_{\chi_y}(f) = \int_{\mathbb{R}} f(x)e^{-2\pi ixy}\, d\mu(x) = \hat{f}(y),$$

the **Fourier transform**.

Observation 11. *The Fourier transform is a special case of the Gel'fand transform.*

As it turns out, so is the Laplace transform!

4.2 C*-algebras and the Gel'fand-Naimark theorem

We take a little look into the future after the Gel'fand transform. Collaborating with Mark Naimark (1909–1978), Gel'fand decided (much like von Neumann) to consider involution structures.

An **involution** on a normed algebra $\mathcal{A}$ is a map $x \mapsto x^*$ satisfying for all $x, y \in \mathcal{A}$:

(i) $*$ is conjugate linear.

(ii) $(x^*)^* = x$.

(iii) $(xy)^* = y^*x^*$.

(iv) $\|x^*\| = \|x\|$.

By (iv), an involution is always continuous.

If $\mathcal{A}$ has an involution, we call it **involutive** or a ***-algebra**. If in addition the involution satisfies

$$\|x^*x\| = \|x\|^2, \quad x \in \mathcal{A},$$

then we say that $\mathcal{A}$ is a C^*-algebra.

Example 4.3. If X is compact, then $C(X)$ is a commutative C^*-algebra with the supremum norm and the involution given by complex conjugation. As well, the space $B(H)$ of bounded linear operators on a Hilbert space H is a C^*-algebra with the operator norm and the involution given by the adjoint operator.

An element x of a C^*-algebra is called **self-adjoint** if $x^* = x$. It turns out that the spectrum of a self-adjoint element is necessarily real (one might have guessed this!). A ***-homomorphism** is an algebra homomorphism which commutes with the involution. *The key result of Gel'fand and Naimark in 1943 was the following eponymous theorem which characterized the behaviour of the Gel'fand transform for *-algebras:*

It's a fact of C^*-algebras that any *-homomorphism has norm $\leqq 1$ and that any injective *-homomorphism is in fact a *-isomorphism.

Theorem 4.4 (Gel'fand-Naimark)**.** *Suppose that $\mathcal{A}$ is a commutative unital Banach *-algebra. Then the Gel'fand transform Γ is an isometric *-isomorphism if and only if $\mathcal{A}$ is a C^*-algebra.*

The Gel'fand-Naimark theorem initiated a new interpretation of Hilbert's spectral theory as it allowed mathematicians to consider C^*-algebras in a more abstract, algebraic context.

Further reading and references

- "*An Invitation to C^*-Algebras*", William Arveson.
- "*A Course in Commutative Banach Algebras*", Eberhard Kaniuth.
- "*History of Functional Analysis*", Jean Dieudonné.
- "*Functional Analysis: Spectral Theory*", V.S. Sunder.
- "*Real and Functional Analysis*", Serge Lang.

Dieudonné is superbly written, and often very witty.