Characteristic classes

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December 30, 2019

Characteristic classes are algebraic invariants for vector bundles which measure the degree of curvature. To discuss characteristic classes, it is first necessary to develop a bit of intuition and machinery (namely, the theory of connections) to discuss curvature. Though there are many routes to discuss the theory of characteristic classes, we choose to use Chern-Weil theory to introduce and explore the theory. Finally we state the Chern-Gauss-Bonnet theorem and apparently lose all will to continue to write anything more.

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1 Connections

1.1 Motivation

If we want to differentiate a vector field $X : \mathbb{R}^n \to \mathbb{R}^n$ in the direction of another vector field $Y : \mathbb{R}^n \to \mathbb{R}^n$ we use the **directional derivative**, defined for $p \in \mathbb{R}^n$ as:

$$(\nabla_Y X)(p) := \lim_{t\to 0} \frac{X(p + tY(p)) - X(p)}{t}.$$

Intuitively (as might be clear from the definition), $(\nabla_Y X)(p)$ gives the infinitesimal rate of change of X in the direction of Y at the point p.

Example 1.1. Consider $\mathbb{R}^2$ with

$$X(x,y) = (-y, x), \qquad Y(x,y) = (1,0).$$

If $p = (x,y) \in \mathbb{R}^2$, then we can compute:

$$\begin{aligned}(\nabla_Y X)(p) &= \lim_{t\to 0} \frac{X((x,y) + tY(x,y)) - X(x,y)}{t} \\ &= \lim_{t\to 0} \frac{(-y, x+t) - (-y, x)}{t} \\ &= \lim_{t\to 0} \frac{(0,t)}{t} \\ &= (0,1).\end{aligned}$$

or in the usual manifold notation with coordinates (x,y),

$$\nabla_Y X = \frac{\partial}{\partial y}.$$

It can be shown that for $f \in C^\infty(\mathbb{R}^n)$:

(i) $\nabla_Y X$ is a vector field.

(ii) $\nabla_{fY} X = f(\nabla_Y X)$.

(iii) $\nabla_Y(fX) = (Y(f))X + f(\nabla_Y X)$.

These properties summarize the "important" aspects of the directional derivative which come to mind when we make use of it. However, this formula only works on $\mathbb{R}^n$ due to two problems in particular:

Namely, homogeneity in the first argument, and Leibniz's product rule in the second.

1. The expression "$p + tY(p)$" does not make sense on an arbitrary manifold as addition of a point to a vector is not defined. This can be fixed relatively easily, though. What we are wanting is just a curve γ which passes through p at $t = t_0$ with the direction given by Y, and the definition would be:

$$(\nabla_Y X)(p) = \lim_{t\to t_0} \frac{X(\gamma(t)) - X(\gamma(t_0))}{t - t_0}.$$

   An example of an alternative to $\gamma = p + tY(p)$ that we can use on an arbitrary manifold with any Y is the flow ϕ_t of Y.

2. Assuming we fixed the prior issue, we now are taking the difference

$$\underbrace{X(\phi_t(p))}_{\in T_{\phi_t(p)}M} - \underbrace{X(p)}_{\in T_p M},$$

the minuands not necessarily lying in the same tangent space. Thus this difference is not well-defined!

The reason why $\mathbb{R}^n$ works so well is due to the fact it is *affine*. This does not mean we have completely lost hope, though. In the way outlined above, the concept of a directional derivative encapsulates a method for associating vectors between tangent spaces.

1.2 Affine connections

We will augment our manifold M with an additional structure in the form of an $\mathbb{R}$-bilinear map,

$$\nabla\colon \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM),$$

defined by the defining properties of the directional derivative, $f \in C^\infty(M)$:

Here, $\Gamma(TM)$ denotes the set of sections of the the tangent bundle (i.e., vector fields) on M.

(i) $\nabla_{fY} X = f(\nabla_Y X)$.

(ii) $\nabla_Y(fX) = (Y(f))X + f(\nabla_Y X)$.

This map ∇ is called an **affine connection** (or just “connection”). The immediate question is whether a connection exists for any manifold. Indeed, this is always possible!

Proposition 1.2. *Any manifold admits an affine connection.*

Proof sketch. First, we can show that if $\sum t_i = 1$, then for a family $\{\nabla^i\}$ of connections, $\nabla := \sum t_i \nabla^i$ is a connection. Because a manifold is locally $\mathbb{R}^n$, we have a locally-defined connection being the standard directional derivative on $\mathbb{R}^n$ (often called the **Euclidean connection**), and so by utilizing partitions of unity, we can sew together the Euclidean connections defined on charts to a global connection.

In $\mathbb{R}^n$, a vector X_p in $T_p\mathbb{R}^n$ is associated to vector $X_q \in T_q\mathbb{R}^n$ by “dragging” X_p along a path γ between p and q, meanwhile keeping the vector $X_{\gamma(t)}$ “parallel” to X_p for all t. In terms of the natural connection ∇ on $\mathbb{R}^n$ (i.e. the directional derivative), we do not want the section X to change in the direction of the section γ'. In other words we want:

$$\nabla_{\gamma'(t)} X_{\gamma(t)} = 0,$$

which adequately describes this parallel dragging.

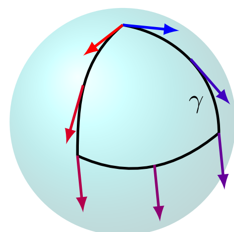
With this observation in mind, a connection ∇ on M specifies a notion of parallelity on M. If $\gamma\colon (a,b) \to M$ is a curve on M, then we say that a section $X\colon \operatorname{im}\gamma \to M$ is **parallel** along γ if

$$\nabla_{\gamma'} X = 0.$$

If a vector $Y_0 \in T_{\gamma(t_0)}M$ extends to a parallel section Y along γ, then Y is the **parallel transport** of $Y_{\gamma(t_0)}$ along γ. Without getting too far into the details, in local coordinates $\nabla_{\gamma'} Y = 0$ and $Y_{\gamma(t_0)} = Y_0$ defines an ordinary differential equation and initial condition which Picard-Lindelöf tells us exists and is unique. Thus ∇ provides us a method for associating vectors between tangent spaces.

In $\mathbb{R}^n$, we may draw any polygon with a vertex p and note that parallel transporting a vector $X_p \in T_p\mathbb{R}^n$ around the loop will yield X_p. That is, dragging

X_p around $\mathbb{R}^n$ and returning back to p leaves the vector unchanged. However, parallel transport is not always as ideal on other manifolds. Consider S^2 and a triangle γ on S^2 which traces the boundary of an octant of S^2. It is easily checked visually that parallel transport of a vector around γ does not preserve the vector.



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of "curvature" on S^2, whereas $\mathbb{R}^n$ is flat. The infinitesimal analogue for this is determining whether ∇_X and ∇_Y commute for parallel sections X and Y. That is, the operator

$$[\nabla_X, \nabla_Y] = \nabla_X \nabla_Y - \nabla_Y \nabla_X$$

measures curvature to some degree. However, there is a small hiccup with using only this operator as the curvature: even on $\mathbb{R}^n$, $[X, Y]$ does not necessarily vanish, despite being flat. In these cases, $[\nabla_X, \nabla_Y]$ will include a possibly non-zero term $\nabla_{[X,Y]}$. To avoid this, and thus force the curvature on $\mathbb{R}^n$ to be zero (and thus "flat"), we define the curvature to be $R(X, Y) \in \operatorname{End}\Gamma(TM)$ by:

$$R(X, Y) := [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}.$$

Parallel transport is commonly said to be the local realization of a connection, whereas the connection is the infinitesimal realization of parallel transport.

A succinct way of defining the curvature is by assigning $R(X, Y)$ as the degree to which $\nabla\colon X \mapsto \nabla_X$ fails to be a Lie algebra homomorphism.

1.3 Connections on vector bundles

The tangent bundle TM is the prototypical example of a "vector bundle", a manifold with a projection which has for each point of M a fibre which is endowed with the structure of a vector space. Recall that a **rank k vector bundle** E over M is a manifold E with an associated smooth projection map $\pi\colon E \twoheadrightarrow M$ such that for each $p \in M$:

(i) where $E_p := \pi^{-1}(p)$ has the structure of a k-dimensional vector space.

(ii) There exists a "trivializing neighbourhood" $U \subseteq M$ of p with the property that:

$$\pi^{-1}(U) \underset{\text{diff.}}{\cong} U \times \mathbb{R}^k,$$

which restricts to the linear isomorphism $E_p \cong \{x\} \times \mathbb{R}^k$.

Given a vector bundle $E \to M$, the manifold M is called the **base space** of the bundle. Denote by $\Gamma(E)$ the set of **sections** of the bundle $\pi\colon E \to M$.

The goal for this section will be to discuss the generalization of connections on manifolds to connections on vector bundles, and instead of focusing on only the tangent bundle of a manifold, we will enjoy the comfort of a more general theory that applies to any vector bundle.

Example 1.3. The simplest example of a vector bundle over a manifold M is the **trivial bundle**, $M \times \mathbb{R}^k$. A trivializing chart is simply the product of a chart on M with the identity on $\mathbb{R}^k$.

Example 1.4. Consider S^1 as the base space, and to each $p \in S^1$, associate a copy of $\mathbb{R}$. Visually, it is easy to come up with two distinct bundles over S^1 in this way: the cylinder, and the Möbius band. These are examples of **line bundles** over S^1.

The Möbius band is not orientable, while the cylinder is, so they are not diffeomorphic.

Exercise 1.5. Is the vector bundle E over S^1 with two twists (i.e. adding another twist to the Möbius band) distinct from these two vector bundles?

Hint. Consider the boundary of E.

Example 1.6. The cotangent bundle T^*M of a manifold is a vector bundle. As well, the k^{th} exterior power $\Lambda^k T^*M$ is also a vector bundle.

Example 1.7. Consider the set

$$L = \{(\ell, v) \in \mathbb{RP}^n \times \mathbb{R}^{n+1} \mid v \in \ell\}.$$

This is called the **tautological line bundle** on $\mathbb{RP}^n$, given the projection $\pi\colon (\ell, v) \mapsto \ell$. As an exercise, try coming up with the trivializing charts for this bundle.

We can make an analogous construction of the tautological *complex* line bundle over $\mathbb{CP}^n$.

A **vector bundle morphism** between $\pi_E\colon E \to M$ and $\pi_F\colon F \to N$ is a pair (φ, Φ) of smooth maps $\varphi\colon M \to N$ and $\Phi\colon E \to F$ such that the following diagram commutes:

$$\begin{array}{ccc} E & \xrightarrow{\Phi} & F \\ {\scriptstyle \pi_E}\downarrow & & \downarrow{\scriptstyle \pi_F} \\ M & \xrightarrow[\varphi]{} & N \end{array} \qquad \text{i.e. } \pi_F \circ \Phi = \varphi \circ \pi_E,$$

and such that for each $p \in M$ we have the restriction of Φ to the fibres, $\Phi_p\colon E_p \to F_{\varphi(p)}$. This gives us several different options for categories depending on what we are interested in, such as the category of vector bundles over a given manifold, of a given rank, or just any real vector bundle at all.

What we have already established is an affine connection which happens to be a structure on the tangent bundle of M. With this inspiration, a **connection** on a vector bundle $\pi\colon E \to M$ is an $\mathbb{R}$-bilinear map:

$$\nabla\colon \Gamma(TM) \times \Gamma(E) \to \Gamma(E),$$

such that for $f \in C^\infty(M)$ we have:

(i) $\nabla_{fX} s = f(\nabla_X s)$.

(ii) $\nabla_X(fs) = (X(f))s + f(\nabla_X s)$.

In this way, any affine connection on M is a connection on the tangent bundle.

Note well that the first argument is still a section of TM (i.e. a vector field), and is not a section of E like the other argument.

Exercise 1.8. Show that if $\nabla^{(1)}$ and $\nabla^{(2)}$ are connections on $E \to M$, then for any $t \in [0,1]$ we have $(1-t)\nabla^{(1)} + t\nabla^{(2)}$ is also a connection on $E \to M$.

This makes connections a **convex** structure. This is ubiquitous in geometry.

We can then use a similar approach to the proof of the existence of an affine connection (utilizing the trivializing neighbourhoods of the vector bundle) to conclude the following:

Proposition 1.9. *Any vector bundle admits a connection.*

The curvature of a connection on a vector bundle is defined analogously to the affine case,

$$R(X,Y) := [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}.$$

Exercise 1.10. Show that the curvature R is alternating in the first two arguments:

$$R(X,Y) = -R(Y,X),$$

and that it is $C^\infty(M)$-multilinear as a map

$$\Gamma(TM) \times \Gamma(TM) \times \Gamma(E) \to \Gamma(E).$$

If $E = TM$, these properties actually make R into a $(1,3)$-**tensor field**: a section of the bundle of tensors of type. $(1,3)$ (contravariant order 1, covariant order 3),

$$T^{(1,3)}M = TM^{\otimes 1} \otimes T^*M^{\otimes 3}.$$

That is, at a point p, R_p is a map

$$R_p\colon T_pM \times T_pM \times T_pM \to T_pM.$$

More generally, if E is any vector bundle, then at a point, it is a map

$$R_p\colon T_pM \times T_pM \to \mathrm{End}(E_p).$$

If we want a local description of a connection, then we would hope that connections behave nicely locally. That is, we want connections to have a notion of restriction to open subsets of M. The concept of "local operators" captures this behaviour.

Let $\varphi\colon \Gamma(E) \to \Gamma(F)$ be an $\mathbb{R}$-linear map where both $E, F \to M$ are vector bundles over M. The map φ is a **local operator** if whenever $s \in \Gamma(E)$ vanishes on an open set U, then so does $\varphi(s)$.

Exercise 1.11. Using bump functions, show that being $C^\infty(M)$-linear is sufficient to be a local operator.

Recall that a **local section** of $\pi\colon E \to M$ over an open subset $U \subseteq M$ is a smooth map $s\colon U \to E$ which exhibits the section property, $\pi \circ s = \mathrm{id}_U$. We denote the set of local sections over U to be $\Gamma(U, E)$. The "local" aspect of local operators arises from how they behave with respect to local sections. Given a local operator $\varphi\colon \Gamma(E) \to \Gamma(F)$, how might we apply φ to a local section $s \in \Gamma(U, E)$?

To be consistent, we also denote $\Gamma(E) := \Gamma(M, E)$.

Exercise 1.12. Using bump functions, show that when given a local section $s \in \Gamma(U, E)$, we can find for any $p \in U$ a global section $\tilde{s} \in \Gamma(E)$ which agrees with s on some neighbourhood $V \subseteq U$ of p.

Assuming you have done your homework, for each $p \in U$, we can find a global section $\tilde{s} \in \Gamma(E)$ which agrees with s on some neighbourhood V of p. Applying φ to $\tilde{s}$ seems to be the obvious course of action, but we have made a choice in picking $\tilde{s}$. Fortunately, since we are only concerned about what happens on U, we can take advantage of φ being a local operator.

Indeed, if $\tilde{\tilde{s}} \in \Gamma(E)$ is another global section which agrees on V with s and $\tilde{s}$, then $\tilde{s} - \tilde{\tilde{s}}$ vanishes on V, so $\varphi(\tilde{s} - \tilde{\tilde{s}})$ vanishes on V using that φ is a local operator. Thus we have that $\varphi(\tilde{s})$ and $\varphi(\tilde{\tilde{s}})$ agree on V. Thus we have a well-defined local section $\varphi|_U(s) \in \Gamma(U, F)$ defined by:

$$\varphi|_U(s)_p := \varphi(\tilde{s})_p.$$

Exercise 1.13. Show that for any $t \in \Gamma(E)$ we have:

$$\varphi|_U(t|_U) = \varphi(t)|_U.$$

Now we can prove that connections are local operators!

Note that they are not $C^\infty(M)$-linear in the second argument, so it is not *that* easy.

Proposition 1.14. *Suppose ∇ is a connection on $E \to M$. If one of $X \in \Gamma(TM)$ or $s \in \Gamma(E)$ vanishes on an open subset $U \subseteq M$, then $\nabla_X s$ does, too.*

Proof. Because ∇ is $C^\infty(M)$-linear in the first argument, it is easily a local operator $\Gamma(TM) \to \Gamma(E)$. So now suppose s vanishes on $U \subseteq M$. For each $p \in U$ we can find a bump function f around p supported in U. On one hand, we note:

$$(\nabla_X(fs))_p = (X_p(f))s_p + f(p)(\nabla_X s)_p = (\nabla_X s)_p,$$

appealing to $X_p(f) = 0$ (since f is locally constant at p). However, by assumption, s vanishes on the support of f, so $fs \equiv 0$, so $\nabla_X(fs) = 0$, and we conclude:

$$(\nabla_X s)_p = 0.$$

■

So as it turns out, connections behave nicely with respect to restriction. We thus can view a connection ∇ restricted to an open subset $U \subseteq M$, seen as a map $\Gamma(U, TM) \times \Gamma(U, E) \to \Gamma(U, E)$.

1.4 Local description of connections

Now that we know for certain that connections behave nicely to restriction, we can consider connections restricted to charts. In particular, we (well at least I am!) interested in a local description of a connection.

First, we (perhaps for some readers, re)visit the analogous concept to bases for vector bundles. If $E \to M$ is a rank k vector bundle, then a **frame** over an open subset $U \subseteq M$ is a collection $e_1, e_2, \ldots, e_k \in \Gamma(U, E)$ such that at each $p \in U$, the collection $e_1(p), e_2(p), \ldots, e_k(p)$ is a basis of E_p. The existence of a frame is equivalent to the existence of a trivialization.

Example 1.15. Consider the tangent bundle TT^2 of the torus, T^2. There are two independent non-vanishing vector fields on the torus (one around the major, and one around the minor circumference). Thus TT^2 admits a global frame and thus is trivial, $TT^2 \cong T^2 \times \mathbb{R}^2$.

On the other hand, the *hairy ball theorem* states that there is no non-vanishing vector field on S^2, and so TS^2 is not trivial.

So picking a trivializing set $U \subseteq M$ for our vector bundle $E \to M$, we automatically have a frame $e_1, e_2, \ldots, e_k$ over U. Any $s \in \Gamma(U, E)$ may be expressed locally on U as:

$$s = \sum_{i=1}^{k} a^i e_i.$$

Thus for a local section $X \in \Gamma(U, TM)$, we may write $\nabla_X s$ locally as:

$$\nabla_X s = \sum_{i=1}^{k} \nabla_X(a^i e_i) = \sum_{i=1}^{k} \left((X(a^i))e_i + a^i(\nabla_X e_i)\right),$$

and so the local description of $\nabla_X s$ depends on our understanding of $\nabla_X e_i$. Being sections themselves, we may write

$$\nabla_X e_i = \sum_{j=1}^{k} \omega_i^j(X) e_j,$$

where the ω_i^j depend on X, acting as $C^\infty(U)$-linear functions $\Gamma(TM, U) \to C^\infty(U)$.

Exercise 1.16. Show that a $C^\infty(M)$-linear function $\Gamma(TM) \to C^\infty(M)$ is a 1-form on M.

Being 1-forms, the ω_i^j are 1-forms on U, called **connection forms**. The matrix $\omega := \left[\omega_j^i\right]$ is called the **connection matrix** of ∇ with respect to the frame $e_1, e_2, \ldots, e_k$.

Analogously, we can repeat this process with the curvature, $R(X,Y)$ and see that its local description depends on understanding $R(X,Y)e_i$, which we may write as:

$$R(X,Y)e_i = \sum_{j=1}^{k} \Omega_i^j(X,Y)e_j.$$

The Ω_i^j inherit bilinearity and alternativity from $R(X,Y)$, and hence can be seen as 2-forms on U, called **curvature forms**. The matrix $\Omega := \left[\Omega_j^i\right]$ is called the **curvature matrix** of ∇ with respect to the frame $e_1, e_2, \ldots, e_k$.

The definition of the curvature, R, depends on the connection, and so there ought to be some dependency of the curvature forms on the connection forms. In what follows we will calculate this relation, but beware—it is not the most pretty derivation! We will take it slowly, though. Recall,

$$R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]},$$

so we first try calculating the first term

$$[\nabla_X, \nabla_Y] = \nabla_X \nabla_Y - \nabla_Y \nabla_X$$

by computing $\nabla_X \nabla_Y e_j$ (the other half will be easy to write down after computing just this term). We start by applying the definition of the connection forms and using linearity of ∇:

$$\nabla_X \nabla_Y e_j = \nabla_X \left(\sum_k \omega_j^k(Y) e_k \right) = \sum_k \nabla_X \left(\omega_j^k(Y) e_k \right),$$

which sets us up to use the Leibniz rule in order to apply the definition of the connection forms again:

$$\begin{aligned}
\sum_k \nabla_X \left(\omega_j^k(Y) e_k \right) &= \sum_k X(\omega_j^k(Y)) e_k + \omega_j^k(Y) \nabla_X e_k \\
&= \sum_k X(\omega_j^k(Y)) e_k + \sum_k \sum_\ell \omega_j^\ell(Y) \omega_\ell^k(X) e_k,
\end{aligned}$$

and so we have the two expressions:

$$\begin{cases} \nabla_X \nabla_Y e_j = \sum_k X(\omega_j^k(Y))e_k + \sum_k \sum_\ell \omega_j^\ell(Y)\omega_\ell^k(X)e_k, \\ \nabla_Y \nabla_X e_j = \sum_k Y(\omega_j^k(X))e_k + \sum_k \sum_\ell \omega_j^\ell(X)\omega_\ell^k(Y)e_k. \end{cases}$$

Noting that $\nabla_{[X,Y]}e_j = \sum \omega_j^k([X,Y])e_k$, we can group terms together to get:

$$\begin{aligned} R(X,Y)e_j &= \sum_k \left(X(\omega_j^k(Y)) - Y(\omega_j^k(X) - \omega_j^k([X,Y])\right) e_k \\ &\quad + \sum_\ell (\omega_j^\ell(Y)\omega_\ell^k(X) - \omega_j^\ell(X)\omega_\ell^k)e_k, \end{aligned}$$

where the fist term can be re-written with the identity for 1-forms α:

If you are unfamiliar with this identity, try proving it! Let $\alpha = f\,dg$ for $f, g \in C^\infty(M)$ and simplify both sides separately.

$$d\alpha(X,Y) = X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X,Y]),$$

and the second term with the definition,

$$(\alpha \wedge \beta)(X,Y) = \alpha(X)\beta(Y) - \alpha(Y)\beta(X),$$

to get:

$$\begin{aligned} R(X,Y)e_j &= \sum_k d\omega_j^k(X,Y)e_k + \sum_\ell (\omega_j^\ell \wedge \omega_\ell^k)(X,Y)e_k \\ &= \sum_k (d\omega_j^k + \sum_\ell \omega_j^\ell \wedge \omega_\ell^k)(X,Y)e_k. \end{aligned}$$

Because $R(X,Y)e_j = \sum \Omega_j^k(X,Y)e_k$, we get the following relation:

Proposition 1.17.

$$\Omega_j^i = d\omega_j^i + \sum_\ell \omega_\ell^j \wedge \omega_i^\ell.$$

Note using matrix notation, the relation can be written as:

$$\Omega = d\omega + \omega \wedge \omega.$$

Exercise 1.18. A quick application of this is one of Bianchi's identities which characterizes the derivative of the curvature matrix:

$$d\Omega = \Omega \wedge \omega - \omega \wedge \Omega.$$

After showing this, try to show more generally that:

$$d(\Omega^n) = \Omega^n \wedge \omega - \omega \wedge \Omega^n.$$

All this we have done *after choosing a frame*. How does a changing the frame affect ω and Ω? We consider this now. Suppose that $\tilde{e}_1, \tilde{e}_2, \ldots, \tilde{e}_k$ is another frame over U, inducing the connection and curvature matrices $\tilde{\omega}$ and $\tilde{\Omega}$. Each $\tilde{e}_i$ can be written in terms of our first frame,

$$\tilde{e}_i = \sum_{j=1}^{k} A_i^j e_j.$$

This gives a change of basis matrix $A = \begin{bmatrix} A_j^i \end{bmatrix}$ which we may view as a map $U \to \mathrm{GL}(k, \mathbb{R})$. Writing frames as row vectors $e := \begin{bmatrix} e_1 & \cdots & e_k \end{bmatrix}$ and $\tilde{e} = \begin{bmatrix} \tilde{e}_1 & \cdots & \tilde{e}_k \end{bmatrix}$, we have:

$$\tilde{e} = eA.$$

With this notation, we can write:

$$\nabla_X e = e\omega(X),$$

and so we can view ∇e as a function $X \mapsto e\omega(X)$, writing:

$$\nabla e = e\omega.$$

To determine how a change of frame affects the connection matrix, we can compute $\nabla \tilde{e}$ in terms of e by way of the relation $\tilde{e} = eA$:

$$\nabla \tilde{e} = \nabla(eA) = (\nabla e)A + e\, dA = e\omega A + e\, dA.$$

Note that the Leibniz rule for ∇ states that

$$\nabla_X(fs) = (X(f))s + f(\nabla_X s).$$

Because $df(X) := X(f)$, the rule is equivalently

$$\nabla_X(fs) = df(X)s + f\nabla_X s.$$

We will want to write this in the form of $\tilde{e}B$ in order to deduce an expression for $\tilde{\omega}$. Because $e = \tilde{e}A^{-1}$, we have:

$$e\omega A + e\, dA = \tilde{e}A^{-1}\omega A + \tilde{e}A^{-1}\, dA = \tilde{e}(A^{-1}\omega A + A^{-1}\, dA),$$

and thus because $\nabla \tilde{e} = \tilde{e}\tilde{\omega}$, we have shown:

$$\tilde{\omega} = A^{-1}\omega A + A^{-1}\, dA.$$

Analogously, considering the curvature R, we see that for a frame e which induces the curvature form Ω,

$$Re = e\Omega.$$

You can use Proposition 1.17 and the equation for the connection matrix to deduce an equation for the curvature matrix.

Carrying out a similar computation, we see:

$$R\tilde{e} = ReA = e\Omega A = \tilde{e}A^{-1}\Omega A,$$

and because $R\tilde{e} = \tilde{e}\tilde{\Omega}$, we have shown:

$$\tilde{\Omega} = A^{-1}\Omega A.$$

Take a moment and just appreciate how pretty that is. Sure, the change of frame equation for the connection form is simple too, but the equation for the curvature matrix remarkably indicates that a change of frame acts on Ω by conjugation by A. The rest of these notes hinges on this fact.

2 Chern-Weil theory

2.1 The Chern-Weil homomorphism

A polynomial p on $\mathrm{gl}(r, \mathbb{R})$ is called **invariant** if for all $A \in \mathrm{GL}(r, \mathbb{R})$,

$$p(A^{-1}XA) = p(X).$$

We denote the set of invariant polynomials on $\mathrm{gl}(r, \mathbb{R})$ by $\mathrm{Inv}(\mathrm{gl}(r, \mathbb{R}))$.

Example 2.1. Consider $\det(X - \lambda I)$ where λ is an indeterminate and X an $r \times r$ matrix of indeterminates. We may write:

$$\det(X + \lambda I) = f_0(X) + f_1(X)\lambda + \ldots + f_{k-1}(X)\lambda^{r-1} + \lambda^r,$$

whose coefficients f_i are polynomials in r^2 many variables. Note that:

$$\det(A^{-1}XA + \lambda I) = \det(A^{-1}(X + \lambda I)A) = \det(X + \lambda I),$$

and so $f_i(A^{-1}XA) = f_i(X)$ as a result. Hence the f_i are invariant polynomials. We call them the **coefficients of the characteristic polynomial** of $-X$.

Example 2.2. If X is an $r \times r$ matrix of indeterminates and $n \geqq 1$, then define

$$\Sigma_n(X) := \mathrm{tr}(X^n),$$

which is a polynomial in r^2 many variables. Because trace is conjugate invariant, $\Sigma_n(X)$ is an invariant polynomial. We call the resulting family of polynomials the **trace polynomials**.

These two families of polynomials are not just examples, but extremely useful to the theory of invariant polynomials due to the following theorem (which we will not prove):

Theorem 2.3. *Let f_n be the characteristic polynomials and Σ_n the trace polynomials. Then:*

$$\begin{aligned}\mathrm{Inv}(\mathrm{gl}(r, \mathbb{R})) &= \mathbb{R}[f_0, f_1, \ldots, f_{r-1}] \\ &= \mathbb{R}[\Sigma_1, \Sigma_2, \ldots, \Sigma_r].\end{aligned}$$

Thus to prove any $\mathbb{R}$-linear property of invariant polynomials, it suffices to prove it on a generating set of either $\{f_0, \ldots, f_{r-1}\}$ or $\{\Sigma_1, \ldots, \Sigma_r\}$.

Let ∇ be a connection on a rank r vector bundle $E \to M$. Fix an open set $U \subseteq M$ with frame e and curvature matrix Ω. We have seen that under a change of coordinates, a curvature matrix is acted on by conjugation. Thus if we have $p \in \text{Inv}(\text{gl}(r, \mathbb{R}))$, then $p(\Omega)$ is independent of choice of frame because if we did change the frame, it would act on Ω by conjugation by some A. But p is invariant and thus:

$$p(A^{-1}\Omega A) = p(\Omega).$$

Hence $p(\Omega)$ is a differential form (if p is homogeneous and of degree k, then $p(\Omega)$ is a $2k$-form) on U which is invariant under change of frame.

This is especially nice with regards to extending $p(\Omega)$ off of U. Indeed, if we take a covering $(U_\alpha)_{\alpha \in I}$ of trivializing charts on M, each with an associated curvature form Ω_α (with respect to a frame e_α), then on any overlap $U_\alpha \cap U_\beta$, $p(\Omega_\alpha)$ and $p(\Omega_\beta)$ agree. Thus $p(\Omega)$ extends uniquely to a *globally defined* differential form on M!

So for each $p \in \text{Inv}(\text{gl}(r, \mathbb{R}))$ we may associate a natural differential form $p(\omega) \in \Omega(M)$. In what follows, we will show that the cohomology class of $p(\Omega)$ is both meaningful and useful. To even belong to a class, $p(\Omega)$ must be closed so we demonstrate this first.

Exercise 2.4. If A and B are square matrices of forms of degrees a and b respectively, then:

(i) $\text{tr}(A \wedge B) = (-1)^{ab} \text{tr}(B \wedge A)$.

(ii) $d \,\text{tr}\, A = \text{tr}\, dA$.

Using these observations, if $n \geqq 1$, then:

$$d\,\text{tr}(\Omega^n) = \text{tr}(d(\Omega^n)) = \text{tr}(\Omega^n \wedge \omega - \omega \wedge \Omega^n) = \text{tr}(\Omega^n \wedge \omega) - \text{tr}(\omega \wedge \Omega^n) = 0,$$

and thus the forms $\Sigma_n(\Omega)$ (associated to the trace polynomials) are all closed forms. By Theorem 2.3, any such form $p(\Omega)$ is closed! Thus $p(\Omega)$ belongs to some cohomology class $[p(\Omega)] \in \Omega^*_{\text{dR}}(M)$.

As it stands, to define Ω we need to choose a connection ∇, and so the cohomology class $[p(\Omega)]$ seems like it depends on ∇. This would be inconvenient as it limits the scope of the theory of these particular cohomology classes to vector bundles with fixed connections, rather than vector bundles in general.

Due to the convexity of connections (see Exercise 1.8), we have any two connections $\nabla^{(0)}$ and $\nabla^{(1)}$ on $E \to M$, are connected by a path of connections $\nabla^{(t)} := (1-t)\nabla^{(0)} + t\nabla^{(1)}$, $t \in [0, 1]$. We will use this fact to show that the cohomology class of $[p(\Omega)]$ does not depend on choice of connection.

For each $t \in [0, 1]$, $\nabla^{(t)}$ induces connection and curvature forms ω_t and Ω_t.

Exercise 2.5. Check that $\omega_t = (1-t)\omega_0 + t\omega_1$.

After checking this, we get that $(\omega_t)_{t\in[0,1]}$ depends smoothly on t. We can write Ω_t in terms of ω_t (Proposition 1.17), and thus Ω_t also depends smoothly on t.

Alike to the proof of the polynomials $p(\Omega)$ being closed, we will rely on the trace polynomials $\Sigma_n(\Omega) = \mathrm{tr}(\Omega^n)$ generating $\mathrm{Inv}(\mathrm{gl}(r,\mathbb{R}))$. Indeed, if the $\Sigma_n(\Omega)$ do not depend on the choice of a connection, then certainly a general $p(\Omega)$ does not, either. Hypothetically, if $\frac{d}{dt}\,\mathrm{tr}(\Omega_t^n) = d\alpha$ for some globally defined differential form α, then integrating both sides, we get:

Here we use that d commutes with $\int \cdot\, dt$. This is equivalent to differentiation under the integral sign. If you bug me enough I will include a proof, otherwise I am too lazy to show you.

$$\mathrm{tr}(\Omega_1^n) - \mathrm{tr}(\Omega_0^n) = \int_0^1 \frac{d}{dt}\,\mathrm{tr}(\Omega_t^n)\,dt = \int_0^1 d\alpha\,dt = \underbrace{d\int_0^1 \alpha\,dt}_{\text{global exact form}} .$$

Passing to the cohomology classes, any exact form is zero and thus we would have

$$[\mathrm{tr}(\Omega_1^n)] = [\mathrm{tr}(\Omega_0^n)],$$

and so $[\Sigma_n(\Omega)]$ would not depend on the choice of connection! Of course, this strategy hinges on finding this nice α, so we set out to do just that.

Exercise 2.6. Show that:

$$\frac{d}{dt}\,\mathrm{tr}\,\alpha = \mathrm{tr}\left(\frac{d\alpha}{dt}\right).$$

In what follows, we suppress the t index of Ω_t (writing simply Ω) and utilize a dot for the derivative of a form with respect to t:

$$\begin{aligned}\frac{d}{dt}\,\mathrm{tr}(\Omega^n) &= \mathrm{tr}\left(\frac{d\Omega^n}{dt}\right)\\ &= \mathrm{tr}(\dot\Omega\wedge\Omega^{n-1} + \Omega\wedge\dot\Omega\wedge\Omega^{n-2} + \ldots + \Omega^{n-1}\wedge\dot\Omega)\\ &= n\,\mathrm{tr}(\Omega^{n-1}\wedge\dot\Omega),\end{aligned}$$

and then we note that $\Omega = d\omega + \omega\wedge\omega$ (suppressing the t in ω_t), so:

$$\dot\Omega = \frac{d}{dt}(d\omega + \omega\wedge\omega) = d\dot\omega + \dot\omega\wedge\omega + \omega\wedge\dot\omega,$$

so continuing,

$$\begin{aligned}
\frac{d}{dt}\operatorname{tr}(\Omega^n) &= n\operatorname{tr}(\Omega^{n-1}\wedge\dot{\Omega}) \\
&= n\operatorname{tr}(\Omega^{n-1}\wedge(d\dot{\omega}+\dot{\omega}\wedge\omega+\omega\wedge\dot{\omega})) \\
&= n\operatorname{tr}(\Omega^{n-1}\wedge d\dot{\omega}+\Omega^{n-1}\wedge\dot{\omega}\wedge\omega+\Omega^{n-1}\wedge\omega\wedge\dot{\omega}) \\
&= n\operatorname{tr}(\Omega^{n-1}\wedge d\dot{\omega}-\omega\wedge\Omega^{n-1}\wedge\dot{\omega}+\Omega^{n-1}\wedge\omega\wedge\dot{\omega}) \\
&= n\operatorname{tr}(\Omega^{n-1}\wedge d\dot{\omega}+(\Omega^{n-1}\wedge\omega-\omega\wedge\Omega^{n-1})\wedge\dot{\omega}) \\
&= n\operatorname{tr}(\Omega^{n-1}\wedge d\dot{\omega}+d(\Omega^{n-1})\wedge\dot{\omega}), \quad \text{(Exercise 1.18)} \\
&= n\operatorname{tr}(d(\Omega^{n-1}\wedge\dot{\omega})) \\
&= d\left(n\operatorname{tr}(\Omega^{n-1}\wedge\dot{\omega})\right), \quad \text{(Exercise 2.4)}.
\end{aligned}$$

Thus we indeed have that $\frac{d}{dt}\operatorname{tr}(\Omega_t^n)$ is an exact form, but a problem is that it might depend on the neighbourhood (since Ω_t is given by a choice of frame). Luckily, we can repeat the same trick we used to show that $p(\Omega)$ could be pieced together to a globally defined differential form on M.

Exercise 2.7. Show that $n\operatorname{tr}(\Omega^{n-1}\wedge\dot{\omega})$ can be extended to a globally defined differential form on M.

Now our argument is more or less complete for the following proposition:

Proposition 2.8. *The cohomology class of $\Sigma_n(\Omega)=\operatorname{tr}(\Omega^n)$ is independent of connection.*

Proof. Suppose that $\nabla^{(0)}$ and $\nabla^{(1)}$ arre connections on $E\to M$, a vector bundle of rank r. Define $\nabla^{(t)}:=(1-t)\nabla^{(0)}+t\nabla^{(1)}$ which is a connection and induces a connection matrix ω_t and curvature matrix Ω_t. As shown above,

$$\frac{d}{dt}\operatorname{tr}(\Omega_t^n)\,dt = d(n\operatorname{tr}(\Omega_t^{n-1}\wedge\dot{\omega})),$$

and integrating both sides yields:

$$\operatorname{tr}(\Omega_1^n)-\operatorname{tr}(\Omega_0^n) = d\int_0^1 n\operatorname{tr}(\Omega_t^{n-1}\wedge\dot{\omega})\,dt,$$

noting that the integrand is globally defined (you *did* do your homework, didn't you?) the right hand is an globally defined exact form and hence passing to cohomology classes we deduce:

$$[\mathrm{tr}(\Omega_0^n)] = [\mathrm{tr}(\Omega_1^n)].$$

Thus $\Sigma_n(\Omega_0)$ and $\Sigma_n(\Omega_1)$ are cohomologous. ■

Corollary 2.9. *Because* $\mathrm{Inv}(\mathrm{gl}(r,\mathbb{R})) = \mathbb{R}[\Sigma_1,\ldots,\Sigma_r]$*, if* $p \in \mathrm{Inv}(\mathrm{gl}(r,\mathbb{R}))$*, then the cohomology class of* $p(\Omega)$ *is independent of connection.*

Let us just stop and appreciate what we have just shown so far:

1. A rank r vector bundle $E \to M$ admits a connection ∇ which induces a notion of curvature, R.
2. Locally, R is captured as a matrix of 2-forms, the curvature matrix Ω.
3. Under a change of frame, Ω changes by conjugation.
4. An invariant polynomial $p \in \mathrm{Inv}(\mathrm{gl}(r,\mathbb{R}))$ induces a globally defined differential form $p(\Omega)$, independent of choice of frame.
5. The form $p(\Omega)$ is closed and hence belongs to a de Rham cohomology class.
6. Moreover, the cohomology class of $p(\Omega)$ is independent of ∇.

One remarkable thing (but not certainly the most or only remarkable thing) is that the cohomology classes $[p(\Omega)]$ are derived from curvature, and thus it seems they ought to provide information about the curvature. However, the notion of curvature we defined is defined in terms of the connection, ∇! So if $[p(\Omega)]$ *does* measure some aspect of curvature, then because it is independent of ∇ it must be
what we have developed so far is the following theorem:

Theorem 2.10 (Chern-Weil)**.** *Let E be a rank r vector bundle $E \to M$ with a connection ∇ and induced curvature matrix Ω. Then the map*

$$\mathcal{C}_E\colon\ \mathrm{Inv}(\mathrm{gl}(r,\mathbb{R})) \to H^*(M)$$

defined by

$$\mathcal{C}_E(p) := [p(\Omega)],$$

*is an algebra homomorphism, called the **Chern-Weil homomorphism**.*

Recall that an $\mathbb{R}$-algebra is a vector space V endowed with a bilinear product. An algebra homomorphism preserves the products of the algebras, and in particular for $\mathcal{C}_E$, we have

$$\mathcal{C}_E(pq) = [p(\Omega) \wedge q(\Omega)].$$

The elements of $\mathrm{im}\,\mathcal{C}_E$ are called **characteristic classes**, and the forms $p(\Omega)$ are called the **characteristic forms**.

2.2 Categorical interpretation

Let $f: N \to M$ be a smooth map. If $\pi: E \to M$ is a rank r vector bundle over M, then we can *pull back* E to a bundle over N. To o this, we define

$$f^*E := \{(p, v) \in N \times E \mid f(p) = \pi(v)\},$$

that is, f^*E consists of all pairs $(p, v) \in N \times E$ such that $v \in \pi^{-1}(f(p))$. Associated to f^*E are two projections,

$$\pi_1: f^*E \to N, \qquad \pi_2: f^*E \to E.$$

The bundle projection of f^*E is projection to the first factor, π_1. Any trivializing neighbourhood (U, φ) of E pulls back to a trivializing neighbourhood $(f^{-1}(U), \psi)$ of f^*E, where ψ: is defined by

$$\psi(p, v) = (p, \pi_2(\varphi(v))).$$

This bundle is called the **pullback bundle** of E. Any section $s \in \Gamma(E)$ induces a section $f^*s \in \Gamma(f^*E)$ defined by precomposition:

This construction is equivalent to the categorical pullback.

$$f^*s := s \circ f.$$

Exercise 2.11. Verify the details that the pullback bundle f^*E is indeed a vector bundle.

If ∇ is a connection on $E \to M$, it induces a connection matrix ω_e relative to the local frame e over $U \subseteq M$. If $\tilde{e}$ is another frame such that $\tilde{e} = eA$, then we have seen that the connection form induced by $\tilde{e}$ is:

$$\omega_{\tilde{e}} = A^{-1}\omega_e A + A^{-1}dA.$$

We can then compute the pullback of $\omega_{\tilde{e}}$ to a matrix of 1-forms on $f^{-1}(U)$:

$$f^*(\omega_{\tilde{e}}) = (f^*A)^{-1} f^*(\omega_e) f^*A + (f^*A)^{-1} df^*A.$$

Because $\tilde{e} = eA$ implies $f^*\tilde{e} = (f^*e)(f^*A)$ (and because connection matrices are sufficient to describe connections locally), ∇ induces a unique connection on f^*E with connection matrix $f^*(\omega_e)$, relative to the frame f^*e on $f^{-1}(U)$. We will denote this pullback connection by $f^*\nabla$.

The pullback connection $f^*\nabla$ has connection frame $f^*(\omega_e)$ relative to f^*e, and so the induced curvature form on f^*E is given by:

$$d(f^*(\omega_e)) + f^*(\omega_e) \wedge f^*(\omega_e) = f^*(d\omega_e + \omega_e \wedge \omega_e) = f^*\Omega_e.$$

Exercise 2.12. Show that for all $p \in \mathrm{Inv}(\mathrm{gl}(r, \mathbb{R}))$ we have:

$$p(f^*(\Omega_e)) = f^*(p(\Omega_e)).$$

This sets us up with a commutative diagram:

$$\begin{array}{ccc} \mathrm{VBun}_r\, N & \xrightarrow{p} & H^*(N) \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle f^*} \\ \mathrm{VBun}_r\, M & \xrightarrow[p]{} & H^*(M) \end{array}$$

Thus each invariant polynomial p induces a map associated to a manifold M,

$$\mathcal{C}_M \colon \left\{\substack{\text{isomorphism classes} \\ \text{of v.bundles over } M}\right\} \to H^*(M),$$

and hence induces a natural transformation $\mathcal{C}$ between the functors VBun_r and H^*. This provides an alternative definition of characteristic classes, the natural transformation itself being called a characteristic class (associated to p).

2.3 An application of Riemannian bundles

A **Riemannian metric** on M is an assignment of an inner product $g_p \colon T_pM \times T_pM \to \mathbb{R}$ on T_pM to each $p \in M$ such that if $X, Y \in \Gamma(TM)$, then $g(X, Y)$ is a smooth function on M defined by

$$p \mapsto g_p(X_p, Y_p).$$

The local existence problem of metrics is dead simple (an exercise in linear algebra), and noting that a finite sum of metrics is again a metric, a partition of unity makes quick work of the global existence problem.

With the use of metrics, one can interpret lengths and angles, and in particular, with a metric comes a notion of orthogonality. We extend the notion of metrics to vector bundles. A **Riemannian metric** on a vector bundle $E \to M$ is likewise an assignment of an inner product g_p to the fibre E_p for each $p \in M$ such that if $s, t \in \Gamma(E)$, then $g(s, t)$ is a smooth function on M. We call a vector bundle with a metric a **Riemannian bundle**. We define the **length** of a $v \in E_p$ to be

$$\|v\| := \sqrt{g_p(v, v)},$$

and as one might guess from $g_p(v, w)$ one can deduce an angle between $v, w \in E_p$. From this, we have the concept of **orthogonality** when $g_p(v, w) = 0$. This extends easily to the concept of **orthonormality** for subsets of E_p.

Exercise 2.13. Show that any vector bundle admits a Riemannian metric.

Given a Riemannian bundle $E \to M$ with metric g, a connection ∇ is compatible with g if for all $X \in \Gamma(TM)$ and $s, t \in \Gamma(E)$, we have:

$$X(g(s,t)) = g(\nabla_X s, t) + g(s, \nabla_X t).$$

This is yet another analogue of the Leibniz rule.

Exercise 2.14. We say that sections $s, t \in \Gamma(E)$ are **parallel** if at each $p \in M$, $g_p(s_p, t_p) = \|s_p\|\|t_p\|$. Show that a connection ∇ is compatible with g if and only if $g(s,t)$ is constant for any two parallel sections $s, t \in \Gamma(E)$ along a curve.

Exercise 2.15. Show that any vector bundle admits a metric connection.

Suppose that ∇ is a metric connection on a vector bundle $E \to M$. Instead of picking any old frame, let us choose an *orthonormal* frame, e. Note that:

$$\begin{aligned} X(g(e_i, e_j)) &= g(\nabla_X e_i, e_j) + g(e_i, \nabla_X e_j) \\ &= g\left(\sum_k \omega_i^k(X) e_k, e_j\right) + g\left(e_i, \sum_k \omega_j^k(X) e_k\right) \\ &= \omega_i^j(X) + \omega_j^i(X), \end{aligned}$$

and so if $i \neq j$ we get $\omega_i^j = -\omega_j^i$ and $\omega_i^i = 0$. Thus the connection matrix ω is skew-symmetric.

Exercise 2.16. Show that if ω is skew-symmetric, then so is Ω.

What is convenient about metric connections is that we have that the connection and curvature matrices are predictable, *and for our purposes*, characteristic classes are independent of choice of connection. So using a metric connection as a representative connection might yield some easier proofs for more general theorems about characteristic classes.

Exercise 2.17. Show that if a curvature matrix Ω is skew-symmetric, then Ω^{2n} is symmetric and Ω^{2n+1} is skew-symmetric.

Hint. What is $(\Omega \wedge \Omega)^\top$?

Automatically, this observation gives that $\Sigma_{2n+1}(\Omega) := \mathrm{tr}(\Omega^{2n+1}) = 0$ for all n, and thus their associated characteristic classes are also zero. Even though we picked a metric connection, this class is independent of this fact and is zero no matter the connection. Even more, a homogeneous polynomial of odd degree has an odd trace polynomial Σ_{2n+1} dividing each monomial term. This gives the following result:

Proposition 2.18. *If $p \in \mathrm{Inv}(\mathrm{gl}(r, \mathbb{R}))$ is a homogeneous polynomial of odd degree n, then $[p(\Omega)] = 0$ in $H^{2n}(M)$.*

Thus as a set of generators of $H^*(M)$, we have the even trace polynomials!

3 Characteristic classes

3.1 Pontryagin classes

Recall the invariant polynomials f_n which are the coefficients of the characteristic polynomial of $-X$. The n^{th} Pontryagin class of $E \to M$ is defined to be:

Note that we can show that f_{2n+1} is of odd degree, much like Σ_{2n+1}, and thus they correspond

$$p_n(E) := \left[f_{2n}\left(\frac{i}{2\pi}\Omega\right)\right] \in H^{4n}(M).$$

The factor of $1/2\pi$ in $p_n(E)$ is there in order to force there to be a representative of $p_n(E)$ which when integrated over a compact oriented submanifold gives an integer. Note that if $E \to M$ is rank r, then:

$$\det((i/2\pi)\Omega + I) = 1 + p_1(E) + p_2(E) + \ldots + p_{\lfloor r/2 \rfloor}(E),$$

which is called the **total Pontryagin class** of E, denoted by $p(E)$.

Now let M be a closed, orientable $4n$-dimensional manifold and suppose $k_1, \ldots, k_m$ are positive integers who sum to n. Then the Pontryagin number $P_{k_1,\ldots,k_m}(M)$ is defined as:

$$P_{k_1,\ldots,k_m}(M) := \int_M p_{k_1}(TM) \ldots p_{k_m}(TM).$$

One of the purposes of Pontryagin numbers is to vanish on a manifold that is a "boundary". We say that two oriented manifolds M_1 and M_2 are **cobordant** if there exists an oriented manifold N such that:

The notation $M_1 - M_2$ means that union of of M_1 and M_2, where M_2 is given the opposite orientation. Much like how $\{0\}$ and $\{1\}$ are cobordant via $N = [0, 1]$.

$$\partial N = M_1 - M_2.$$

With this definition, M is a boundary if and only if it is cobordant to the empty set. Note that in this case, if a compact, oriented $4n$-manifold $M = \partial N$ then:

$$P_{k_1,\ldots,k_m}(M) = \int_{\partial N} p_{k_1}(TM) \ldots p_{k_m}(TM) = \int_N d(p_{k_1}(TM) \ldots p_{k_m}(TM)) = 0.$$

So all Pontryagin numbers of M vanish when it is a boundary! Thus with regards to cobordism we get the following theorem:

Theorem 3.1. *If M_1 and M_2 are compact, oriented, cobordant manifolds, then their Pontryagin numbers agree.*

Proof. Because they are cobordant, there is an oriented N such that $\partial N = M_1 - M_2$. Thus Pontryagin numbers of $M_1 - M_2$ vanish, and thus since we have $\int_{M_1 - M_2} \alpha = \int_{M_1} \alpha - \int_{M_2} \alpha$, the conclusion follows. ∎

3.2 The Euler class

We now consider the case of oriented vector bundles. An **orientation** on a rank r vector bundle $E \to M$ is an equivalence class of nowhere vanishing sections of the line bundle $\Lambda^r E$ under the equivalence relation:

$$s \sim t \iff t = fs, \quad f > 0.$$

Proposition 3.2. *A rank r vector bundle $E \to M$ has an orientation if and only if the line bundle $\Lambda^r E$ is trivial.*

A frame over $U \subseteq M$ is **positively oriented** if at each point, the frame agrees with the orientation of E_p. Now instead of just discussing orthonormal frames, we can discuss *positively oriented* orthonormal frames.

Given two positively oriented, orthonormal frames e and $\tilde{e}$ over U, there is an $A \colon U \to \mathrm{SO}(r)$ such that:

$$\tilde{e} = eA.$$

So now we focus instead of on $\mathrm{GL}(r, \mathbb{R})$-invariant polynomials, on the more general $\mathrm{SO}(r)$-invariant polynomials. As it turns out, if r is odd, then these sets coincide. If r is even, then we have a new generator to consider.

If $X \in \mathrm{so}(2n)$, the **Pfaffian** of X is the polynomial $\mathrm{Pf}\, X$ satisfying:

$$\det X = (\mathrm{Pf}\, X)^2.$$

Proposition 3.3.

$$\mathrm{Pf}(A^T X A) = \det A \cdot \mathrm{Pf}\, X.$$

This implies that for $A \in \mathrm{SO}(2n)$:

$$\mathrm{Pf}(A^{-1}XA) = \mathrm{Pf}(A^T XA) = \det A \cdot \mathrm{Pf}\, X = \mathrm{Pf}\, X,$$

and so $\mathrm{Pf}\, X$ is $\mathrm{SO}(2n)$ invariant! As in the case for $\mathrm{Inv}(\mathrm{gl}(r, \mathbb{R}))$, we can show that $\mathrm{Pf}(\Omega)$ is a closed, global form which is independent of connection, and thus it is a characteristic class (indeed, it satisfies the categorical definition!). We define the **Euler class** of the oriented vector bundle E of rank $2n$ to be:

$$e(E) := \left[\mathrm{Pf}\left(\frac{1}{2\pi} \Omega \right) \right].$$

This class is so-called due to the following theorem which we will not prove:

Theorem 3.4 (Chern-Gauss-Bonnet)**.** *Let M be a compact, oriented Riemannian manifold. Then:*

$$\int_M \mathrm{Pf}\left(\frac{1}{2\pi} \Omega \right) = \chi(M),$$

the Euler characteristic of M.