(Co)tangent things

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Reminder

Recall that a (topological) manifold is a locally Euclidean, second-countable, Hausdorff topological space, M. By locally Euclidean, we mean for each point p in M there exists an open neighbourhood U of p which is homeomorphic to an open subset V of $\mathbb{R}^n$. The pair (U, φ) of the open set and the homemorphism[1] $\varphi\colon U \to V$ is called a **chart**, the collection of charts on M being called an **atlas**. We can discuss the "smooth structure" of M by giving it a maximal smooth atlas (the transition maps between any two charts are smooth in the usual sense of being infinitely differentiable, and this atlas is maximal with respect to inclusion).

[1] A homemorphism is a continuous bijection with a continuous inverse.

A map $F\colon M \to N$ between manifolds is called smooth at $p \in M$ if and only if for any two charts (U, φ) about p (on M) and (V, ψ) about $F(p)$ (on N) such that $F(U) \subseteq V$, we have that

$$\psi \circ F \circ \varphi^{-1}$$

is a smooth map in the usual sense.

Recall also that if (U, φ) is a chart on M and $f \in C^\infty(M)$, and $r^1, \dots, r^n$ are the standard coordinates on $\mathbb{R}^n$, then $x^i := r^i \circ \varphi$ characterize the coordinates on the chart (U, φ). So if $p \in U$, then we define

$$\frac{\partial f}{\partial x^i}(p) := \frac{\partial (f \circ \varphi^{-1})}{\partial r^i}(\varphi(p)),$$

the partial derivative of f at p with respect to x^i. Because of the compatibility of charts, this will not depend on the choice of (U, φ).

The idea now is that we want to study manifolds by linearising them. This is not a strange thing to anyone who has ever taken calculus as this is exactly what is done in a first course in calculus, except only the special case of $\mathbb{R}^n$. That is, we study $\mathbb{R}^n$ by looking at the linearisations of functions (i.e. derivatives) on $\mathbb{R}^n$. In order to do this in the general context of manifolds, we will define and study their tangent bundles.

The main purpose of this overview is to learn about the tangent bundle, it's dual notion the cotangent bundle, differential forms, their integration, and cohomology. We end with calculating the de Rham cohomology which makes use of the theory we develop.

1 The tangent bundle

1.1 Motivation

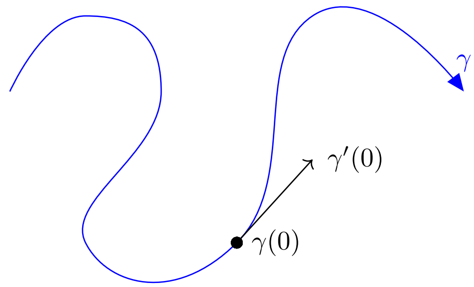
Consider a smooth curve γ in $\mathbb{R}^n$. We will write γ as a function

$$\gamma\colon (-\varepsilon, \varepsilon) \to \mathbb{R}^n \colon t \mapsto (\gamma_1(t), \gamma_2(t), \dots \gamma_n(t)),$$

where we call the functions $\gamma_i\colon (-\varepsilon, \varepsilon) \to \mathbb{R}$ the component functions of γ. The **velocity vector** of γ at time $t = 0$ is given by:

$$\gamma'(0) := (\gamma_1'(0), \gamma_2'(0), \dots, \gamma_n'(0)) \in \mathbb{R}^n, \qquad \gamma_i'(0) := \frac{d\gamma_i}{dt}(0).$$

We can imagine the vector $\gamma'(0)$ to be an arrow tangent to the curve γ, starting at $\gamma(0) \in \mathbb{R}^n$.



With this picture in mind, we write $\gamma'(0)$ as a pair,

$$(\gamma(0), \gamma'(0)) \in \mathbb{R}^n \times \mathbb{R}^n,$$

which is an example of what we will call a **tangent vector at** $p = \gamma(0)$. The set of all possible tangent vectors at a $p \in \mathbb{R}^n$ is called the **tangent space** of $\mathbb{R}^n$ at p and is denoted by $T_p\mathbb{R}^n$. It represents all the possible velocity vectors of smooth curves which pass through p. If we "bundle" these tangent spaces for each $p \in \mathbb{R}^n$ together in a disjoint union, we get the **tangent bundle** of $\mathbb{R}^n$:

$$T\mathbb{R}^n = \bigsqcup_{p \in \mathbb{R}^n} T_p\mathbb{R}^n.$$

In the particular case for $\mathbb{R}^n$, the tangent bundle is trivial, $T\mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n$, and so it may not seem very interesting as of yet (but it will later!).

Observe that there are many distinct curves through a point p that yield the same velocity vector. We have a natural representative (natural in the sense that it is the simplest) smooth curve γ which passes through some point $p = (p_1, p_2, \ldots, p_n) \in \mathbb{R}^n$ with velocity vector $v_p = (v_1, v_2, \ldots, v_n) \in \mathbb{R}^n$ (we use the subscript p to keep track of where our vector is based at):

$$\gamma(t) := (p_1 + v_1 t,\, p_2 + v_2 t,\, \ldots,\, p_n + v_n t),$$

where clearly $\gamma'(0) = v_p$.

Associated to each such representative γ (equivalently, to each v_p) is the directional derivative $\nabla_{v_p} \colon C^\infty(\mathbb{R}^n) \to \mathbb{R}$ at p, defined by:

$$\nabla_{v_p}(f) := \frac{d(f \circ \gamma)}{dt}(0) \in \mathbb{R}.$$

If we apply the chain rule to this, we get:

$$\nabla_{v_p}(f) = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x^i}(p)\, \frac{d\gamma_i}{dt}(0) \right) = \sum_{i=1}^{n} v_i \frac{\partial f}{\partial x^i}(p),$$

where $x^1, x^2, \ldots, x^n$ are the standard coordinate functions[2] in $\mathbb{R}^n$. This coincides with the sometimes-used definition of:

$$\nabla_{v_p}(f) = v_p \cdot \nabla(f)(p),$$

[2]The superscript indices is a, albeit painful, convention.

where $\nabla(f)(p)$ is the gradient of f evaluated at p. Note then that we can just consider ∇_{v_p} as an operator:

$$\nabla_{v_p} = \sum_{i=1}^{n} v_i \frac{\partial}{\partial x^i}(p).$$

This establishes a correspondence $v_p \leftrightarrow \nabla_{v_p}$ which allows us to identify the concrete tangent vectors with a more abstract operator. This correspondence does not depend on the actual particular representative γ. Some particular properties of ∇_{v_p} include:

(i) If $\alpha, \beta \in \mathbb{R}$ and $f, g \in C^\infty(\mathbb{R}^n)$ then:

$$\nabla_{v_p}(\alpha f + \beta g) = \alpha \nabla_{v_p}(f) + \beta \nabla_{v_p}(g).$$

(ii) If $f, g \in C^\infty(\mathbb{R}^n)$ then:

$$\nabla_{v_p}(f \cdot g) = \nabla_{v_p}(f) \cdot g(p) + f(p) \cdot \nabla_{v_p}(g).$$

In summary, ∇_{v_p} is an $\mathbb{R}$-linear map on the vector space $C^\infty(\mathbb{R})$ which satisfies the Leibniz (product) rule .

1.2 The abstract tangent space

Now we generalize the correspondence $v_p \leftrightarrow \nabla_{v_p}$ to arbitrary manifolds using these properties of ∇_{v_p}. First we will define the tangent space of a manifold at a point.

Let M be a smooth manifold and $p \in M$. We define a **tangent vector** to M at p to be a map $v_p \colon C^\infty(M) \to \mathbb{R}$ such that:

(i) If $\alpha, \beta \in \mathbb{R}$ and $f, g \in C^\infty(M)$ then:

$$v_p(\alpha f + \beta g) = \alpha v_p(f) + \beta v_p(g).$$

(ii) If $f, g \in C^\infty(M)$ then:

$$v_p(f \cdot g) = v_p(f) \cdot g(p) + f(p) \cdot v_p(g).$$

The set of all tangent vectors is called the **tangent space** of M at p, and is denoted by T_pM. It not difficult to show that T_pM is an $\mathbb{R}$-vector space[3] and so T_pM is a linearised version of M at the point p. As with the case of $\mathbb{R}^n$, taking the disjoint union of all the tangent spaces gives us what is called the **tangent bundle** of M:

$$TM := \bigsqcup_{p \in M} T_pM.$$

[3] Indeed there are many equivalent ways to define T_pM, but this way is the easiest to see the vector space structure.

Proposition 1.1. *Let M be an smooth n-manifold and $p \in M$. If $(U, \varphi) = (U, x^1, \dots, x^n)$ is a chart about p, then*

$$\left\{ \frac{\partial}{\partial x^i}(p) \,\middle|\, 1 \leqq i \leqq n \right\}$$

is a basis for T_pM.

Proof. First a little lemma:

> Let $V \subseteq \mathbb{R}^n$ be a convex neighbourhood of 0 and $f \colon U \to \mathbb{R}$ a smooth function with $f(0) = 0$. Note that:
>
> $$f(x) = \int_0^1 \frac{d}{dt} f(tx)\, dt,$$
>
> and so by the chain rule,
>
> $$f(x) = \int_0^1 \sum_i \frac{\partial f(tx)}{\partial x^i} x^i \, dt = \sum_i \left(\int_0^1 \frac{\partial f(tx)}{\partial x^i}\, dt \right) x^i,$$
>
> so $f(x) = \sum a_i(x) x^i$ for some smooth a_i. Moreover, note that after differentiating this and evaluating at 0, $a_i = (\partial f / \partial x^i)(0)$.

Now suppose that $(U, \varphi) = (U, x^1, \dots, x^n)$[4] is a chart around $p \in M$ such that $\varphi(p) = 0$. Pick an $f \in C^\infty(M)$ defined on U such that $f(p) = 0$ (if it doesn't, set $f' = f - f(p)$ and continue). By the above lemma, there is a neighbourhood $V \subseteq \varphi(U)$ such that the function $g - f \circ \varphi^{-1}$ has the form $g = \sum g_i r^i$ for smooth g_i such that $g_i(0) = (\partial g / \partial r^i)(0)$. For any $q \in U$

$$f(q) = g(\varphi(q)) = \sum g_i(\varphi(q)) x^i(q),$$

and so if $v_p \in T_pM$, we have:

$$\begin{aligned} v_p(f) &= v_p \left(\sum (g_i \circ \varphi) x^i \right) \\ &= \sum g_i(\varphi(p)) \cdot v_p(x^i) \\ &= \sum v_p(x^i) \cdot \frac{\partial f}{\partial x^i}(0), \end{aligned}$$

as $x^i(p) = 0$ and $g_i(h(p)) = g_i(0) = (\partial(f \circ \varphi^{-1}) / \partial r^i)(0)$. Hence the set of partials is a spanning set. Note that $(\partial(x^j)/\partial x^i)(0) = \delta_{ij}$, and hence the set is linearly independent. ∎

Hence we have that any $v_p \in T_pM$ looks like $v_p = \sum_i v_i \frac{\partial}{\partial x^i}(p)$ for $v_i \in \mathbb{R}$.

[4] We reserve the coordinates r^i for those on $\mathbb{R}^n$

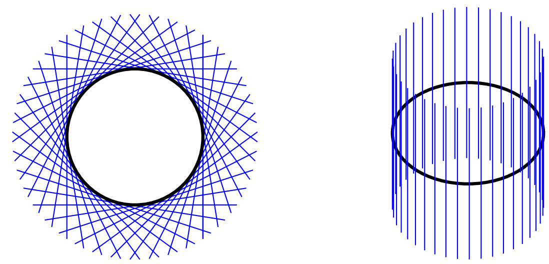
Example 1.2 (Tangent space of S^1)**.** Consider $M = S^1$, the unit circle. Geometrically, it's clear that any velocity vector of a smooth curve at $p \in S^1$ is a tangent vector to S^1 at p. The tangent space T_pS^1 at p is then all possible magnitudes of this tangent vector—that is, $T_pS^1 \cong \mathbb{R}$.

With respect to the abstract formulation of the tangent space, if $(U, \varphi) = (U, \theta)$ is a chart on S^1 around p, then

$$T_pS^1 = \{\lambda(\partial/\partial\theta)(p) \mid \lambda \in \mathbb{R}\} \cong \mathbb{R},$$

just as our geometric intuition suggests!

If we imagine T_pS^1 as the real line $\mathbb{R}$ lying tangent to S^1 at p, then the tangent bundle, TS^1 can then be imagined as a cylinder ($S^1 \times \mathbb{R}$) by rotating these lines around.



1.3 Reconciling the motivation with the abstract

Recall that to motivate tangent vectors, we considered smooth curves in $\mathbb{R}^n$ and then showed a correspondence between velocity vectors $\gamma'(0) = v_p$ of these curves and directional derivatives ∇_{v_p}. This correspondence doesn't fall apart in the case of manifolds, but the curve notion just ends up being unwieldy for the algebraic approach to manifolds that we will be taking.

Consider a smooth curve $\gamma\colon (-\varepsilon, \varepsilon) \to M$ on M such that $\gamma(0) = p$. We identify the notations of $\gamma'(0) = v_p$ and ∇_{v_p} and define for $f \in C^\infty(M)$:

$$\gamma'(0)f := \frac{d(f \circ \gamma)}{dt}(0).$$

It's not difficult to show that $\gamma'(0) \in T_pM$. If we consider a chart $(U, \varphi) = (U, x^1, \ldots, x^n)$ around $p \in M$, we can show that the local-coordinates representation of $\gamma'(0)$ is

$$\gamma'(0) = \sum_{i=1}^{n} \gamma_i'(t) \frac{\partial}{\partial x^i}(0),$$

where $\gamma' := x^i \circ \gamma$, and so in terms of a matrix:

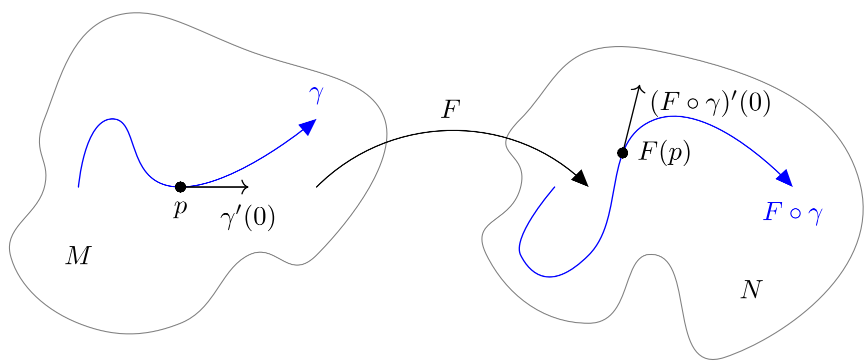
$$\gamma'(0) = \begin{bmatrix} \gamma_1'(0) \\ \gamma_2'(0) \\ \vdots \\ \gamma_n'(0) \end{bmatrix}.$$

Note this is just the usual (as in Section 1.1) "velocity vector" $\alpha'(0)$ of the curve $\alpha := \varphi \circ \gamma$, so it's rather fitting that we use the same notation.

As it turns out, for any $p \in M$ and $v_p \in T_pM$, we can find a curve γ such that $\gamma(0) = p$ and $\gamma'(0) = v_p$. It ends up being an equivalent method to defining the tangent space, but despite its computational perks it lacks the algebraic perks we want to come easy (such the vector space structure on T_pM).

1.4 The pushforward

Consider a smooth map $F\colon M \to N$ between manifolds. A smooth curve γ on M with $\gamma(0) = p$ induces another smooth curve $F \circ \gamma$ on N, and furthermore a tangent vector $v_p := \gamma'(0) \in T_pM$ induces an analogous tangent vector $w_{F(p)} := (F \circ \gamma)'(0) \in T_{F(p)}N$.



We can treat this correspondence $v_p \mapsto w_{F(p)}$ more formally as a function $F_{*,p}\colon T_pM \to T_{F(p)}N$ where we see for $g \in C^\infty(N)$:

$$\begin{aligned}
F_{*,p}(v_p)g &= F_{*,p}(\gamma'(0))g \\
&:= w_{F(p)}(g) \\
&= (F \circ \gamma)'(0)(g) \\
&= \frac{d(g \circ F \circ \gamma)}{dt}(0) \\
&= \gamma'(0)(g \circ F) \\
&= v_p(g \circ F),
\end{aligned}$$

which gives us the more general definition:

$$F_{*,p}(v_p)g := v_p(g \circ F),$$

called the **pushforward** of F at p. When the context is clear, we might choose to write F_* to ease the notation. One can check that:

(i) F_* is a linear map.

(ii) For smooth maps $M \xrightarrow{F} N \xrightarrow{G} P$ between manifolds and $p \in M$:

$$(G \circ F)_{*,p} = G_{*,F(p)} F_{*,p}.$$

(iii) If $\mathrm{id}\colon M \to M$ is the identity, then $\mathrm{id}_*\colon T_pM \to T_pM$ is also the identity.

(iv) If F is a diffeomorphism, then F_* is a linear isomorphism.

So in this process, a smooth $F\colon M \to N$ between curvy manifolds is turned into a linear $F_*\colon T_pM \to T_{F(p)}N$ between linear approximations of manifolds at a point. The result is a "linearisation" of F.

Picking suitable charts $(U, x^1, \ldots, x^m)$ around p and $(V, y^1, \ldots, y^n)$ around $F(p)$ such that $F(U) \subseteq V$, we can express the image of basis of T_pM under F_* in terms the basis of $T_{F(p)}N$. In particular, we know that for suitable $a_j \in \mathbb{R}$,

$$F_*\left(\frac{\partial}{\partial x^i}(p)\right) = \sum_{j=1}^{n} a_j \frac{\partial}{\partial y^j}(F(p)),$$

and application of both sides to the coordinate function y^k yields:

$$\frac{\partial(y^k \circ F)}{\partial x^i}(p) = a_k.$$

Hence, setting $F^k := y^k \circ F$ we get that the matrix representation of F_* in these bases is

$$[F_*] = \left[\frac{\partial F^i}{\partial x^j}\right],$$

the Jacobian matrix of F! So F_* is indeed the manifold-equivalent[5] to the derivative!

1.5 The tangent bundle and vector fields

Let M be a manifold. Define the **tangent bundle** of M to be:

$$TM := \bigsqcup_{p \in M} T_pM,$$

which is associated with the projection $\pi\colon TM \to M\colon v_p \mapsto p$. By gluing the pushforwards $F_{*,p}$ for each $p \in M$, we get a global pushforward, $F_*\colon TM \to TN$. The following theorem we leave unproved.

Theorem 1.3. *If M is a smooth n-manifold, then TM can be given the structure of a smooth $2n$-manifold.*

[5] In general, this is how we generalize things from $\mathbb{R}^n$ to manifolds.

It turns out that the smooth structure that TM inherits is directly from M itself so that the projection π is a smooth map, and so that for any smooth $F\colon M \to N$, the following diagram commutes:

$$\begin{tikzcd} TM \arrow[r, "F_*"] \arrow[d, "\pi"] & TN \arrow[d, "\pi"] \\ M \arrow[r, "F"] & N \end{tikzcd}$$

Moreover, this F_* is a smooth map with respect to the smooth structures of TM and TN.

The thing that is not obvious from our definition is that with respect to the smooth structure, the tangent bundle TM of an n-manifold M is not necessarily diffeomorphic to $M \times \mathbb{R}^n$. It is true that the only two tangent bundles we have defined—for $\mathbb{R}^n$ and S^1—have been diffeomorphic to this "trivial" bundle, but in general it does not hold. If it did, one might guess that the topic of tangent bundles would be really bland and boring!

The idea behind the tangent bundle is that of the more general "vector bundles". A **vector bundle** of rank n is a triple (E, B, π) where:

(i) E and B are topological spaces.

(ii) $\pi\colon E \to B$ is a continuous surjective map.

(iii) For each $x \in B$, $\pi^{-1}(x)$ the structure of an n-dimensional vector space.

(iv) For each $x \in B$ there exists an open neighbourhood U of x such that $\pi^{-1}(U)$ is homeomorphic to $U \times \mathbb{R}^n$.

The final condition states that every point $x \in B$ has a neighbourhood which is "trivial" with respect to the vector bundle—that when you look at $\pi^{-1}(U)$ for a certain U, it *looks like* a "bundle" of copies of $\mathbb{R}^n$, one for each $y \in U$. If a global trivializing neighbourhood $U = B$ exists, then the bundle is called **trivial**.

In our case, we are considering the *smooth* vector bundles $(TM, M, \pi\colon TM \to M)$ where we additionally require smoothness instead of just continuity. An example of where $TM \not\cong M \times \mathbb{R}^n$ is the case for $M = S^2$ (i.e. non-trivial). This is not obvious, but relates to the (non)existence of a non-vanishing vector field and the "hairy ball" theorem[6].

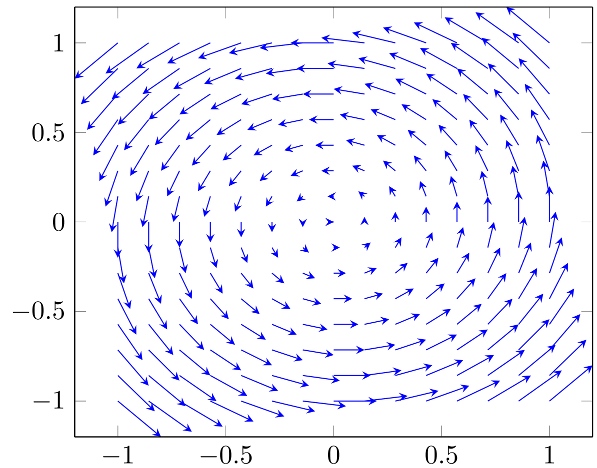
Intuitively, a vector field is a map which takes a point in M and spits out a tangent vector. Hence, a "field of vectors" on M. Formally, a **vector field** is defined to be a smooth map $v\colon M \to TM$ such that $\pi \circ v = \mathrm{id}_M$ (that is, it is a smooth "section" of π).

Example 1.4. Consider $M = \mathbb{R}^2$, defining $v\colon \mathbb{R}^2 \to T\mathbb{R}^2$ by

$$v(x, y) := v_{(x,y)} = (-y, x)$$

[6] One cannot comb a hairy ball without making a cowlick!

which assigns to the point $p = (x, y)$ a vector $v_p = (-y, x)$ (which we draw emanating from point (x, y) on the plane). Pictured below is a portion of $\mathbb{R}^2$ with the vector field v.



On $\mathbb{R}^2 \setminus \{(0,0)\}$ the vector $v_{(x,y)} = (-y, x)$ is non-zero, but at $(0,0)$ however v is $(0,0)$ and we say v vanishes at $p = (0,0)$. Abstractly, without appealing to the diffeomorphism $T\mathbb{R}^2 \cong \mathbb{R}^2 \times \mathbb{R}^2$, this vector field can be written as $v = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$.

This last way of writing v hints at an alternative definition for a vector field. One can define a vector field to be an $\mathbb{R}$-linear map $v\colon C^\infty(M) \to C^\infty(M)$ which satisfies the Leibniz rule,

$$v(f \cdot g) = v(f) \cdot g + f \cdot v(g).$$

In this way for a $p \in M$, $v_p := v(p)$ is "evaluated" in the codomain at p so that it is a map $v_p\colon C^\infty(M) \to \mathbb{R}$, a tangent vector!

2 The cotangent bundle

2.1 Linear algebra review

The usual concept of linear maps can be generalized to the concept of bilinear maps. If V, W, and X are vector spaces, a map $T\colon V \times W \to X$ is **bilinear** if it is linear with respect to each component. That is, if for each $v \in V$ and $w \in W$, we have that $T(v, \cdot)$ and $T(\cdot, w)$ are linear in the usual sense[7].

The tensor product of two vector spaces V, W is a pair $(V \otimes W, \otimes)$ such that:

(a) $V \otimes W$ is a vector space.

[7]This can again be extended to a concept of **multilinearity**

(b) $\otimes \colon V \times W \to V \otimes W$ is a linear map such that for any bilinear map $f \colon V \times W \to X$ (for some vector space X), there exists a unique linear map $\tilde{f} \colon V \otimes W \to X$ such that $f = \tilde{f}\otimes$.

That is, the tensor product $V \otimes W$ "factors" any bilinear maps into the composition of two linear maps in a unique way. The idea is that the tensor product gives us a "free" vector space where bilinear maps $V \times W$ correspond to linear maps.

Theorem 2.1. *The tensor product of V and W exists and is unique.*

Most notably, if $\{v_1, \ldots, v_n\}$ is a basis for V and $\{w_1, \ldots, w_m\}$ is a basis for W, then

$$\{v_i \otimes v_j \mid 1 \leqq i \leqq n, 1 \leqq j \leqq m\}$$

is a basis for $V \times W$. Hence $\dim V \otimes W = \dim V \cdot \dim W$.

Proposition 2.2. *For finite-dimensional real vector spaces U, V, W:*

(i) $V \otimes W \cong W \otimes V$.

(ii) $U \otimes (V \otimes W) \cong (U \otimes V) \otimes W$.

If $\{V_\alpha\}_{\alpha \in I}$ is a family of vector spaces, then the **direct sum**,

$$\bigoplus_{\alpha \in I} V_\alpha,$$

is the vector space consisting of tuples $(a_\alpha)_{\alpha \in I}$ with $a_\alpha \in V_\alpha$ such that all but finitely many of the a_α are zero. The vector space operations are the component-wise operations. With this notation, the **tensor algebra** $T(V)$ is defined to be the algebra:

$$T(V) := \bigoplus_{n \geq 0} V^{\otimes n},$$

where $V^{\otimes 0} := \mathbb{R}$ and $V^{\otimes n} = \underbrace{V \otimes \ldots \otimes V}_{n \text{ times}}$ with the obvious multiplication $(v, w) \mapsto v \otimes w$. This is the free algebra over V. Elements of the tensor algebra look like

$$\sum_{i=0}^{k} a_i v_{i,1} \otimes v_{i,2} \otimes \ldots \otimes v_{i,n_i},$$

for various $v_{i,j} \in V$ and $a_i \in \mathbb{R}$.

What we would now like to do is get the "alternating algebra" where we mimic the property of the cross product that $v \times w = -w \times v$. To this end, we take the quotient[8] of $T(V)$ by the two-sided ideal[9] I generated by the relations

[8] Recall that if W is a linear subspace of a vector space V, then the **quotient space** V/W is the vector space

$$V/W := \{v + W \mid v \in V\},$$

where $v + W = \{v + w \mid w \in W\}$ is the coset of v. This quotient space is a vector space with the operations $\alpha(v + W) := \alpha v + W$ and $(u + W) + (v + W) := (u + v) + W$.

[9] A **two-sided ideal** of the tensor algebra $T(V)$ is a subspace $U \subseteq T(V)$ such that for each $x \in T(V)$ and $u \in U$, we have that $u \otimes x, x \otimes u \in U$.

$v \otimes w + w \otimes v$ for all $v, w \in V$:

$$I := \mathrm{span}\left\{u_1 \otimes \ldots \otimes u_\ell (w_1 \otimes w_2 + w_2 \otimes w_1) \otimes v_1 \otimes \ldots \otimes v_k \mid \forall u_i, v_j, w_1, w_2 \in V\right\}.$$

That is, we end up "identifying" $v \otimes w + w \otimes v$ with 0 which is the same as saying that $v \otimes w = -w \otimes v$. The resulting graded algebra $\Lambda V := T(V)/I$ is the called the **exterior algebra** over V with the natural "exterior" product,

$$v \wedge w := v \otimes w + I.$$

The n^{th} exterior power of V is the n^{th} grade subspace of the exterior algebra,

$$\Lambda^n(V) := \mathrm{span}\{v_1 \wedge \ldots \wedge v_n \mid v_i \in V\}.$$

Because the multiplication $\wedge$ on the exterior algebra makes it into an algebra, we will take for granted that it is a bilinear map.

2.2 The cotangent bundle and 1-forms

Recall that if V is a vector space over $\mathbb{R}$, then the **dual space** of V is the vector space

$$V^* := \{f \colon V \to \mathbb{R} \mid f \text{ is linear}\}.$$

Given that V is finite dimensional with a basis $\{v_1, v_2, \ldots, v_n\}$, there exists a corresponding dual basis $\{f_1, f_2, \ldots, f_n\}$ of V^* such that

$$f_i(v_j) = \delta_{ij}.$$

Because T_pM is a vector space, we can take its dual,

$$T_p^*M := \{\alpha_p \colon T_pM \to \mathbb{R} \mid \alpha_p \text{ is linear}\},$$

which we call its **cotangent space** at $p \in M$. This is also a vector space and we can form the **cotangent bundle** of M:

$$T^*M := \bigsqcup_{p \in M} T_p^*M,$$

which has its associated bundle projection $\pi \colon T^*M \to M \colon \alpha_p \mapsto p$.

With the tangent bundle, we found that the sections of the bundle projection π had the nice description of being vector fields as we know them. In the case of the cotangent bundle, the geometric interpretation of sections of the projection is not so obvious but the sections are certainly no less important!

A smooth section $\alpha \colon M \to T^*M$ of the cotangent bundle is called a differential **1-form** on M. It is our first example of a differential form. We can easily create examples of 1-forms from functions $f \in C^\infty(M)$ by using the "**differential**", d, defined by:

$$df(p)v_p := (df)_p v_p := v_p(f) \in \mathbb{R}.$$

It turns out that the differential is just the pushforward of f in disguise. This is more apparent if we write:

$$f_{*,p}(v_p) = (df)_p(v_p)\frac{d}{dt}(f(p)).$$

This gives an easy identification of the differential and pushforward of f.

Fixing a point $p \in M$ and considering a chart $(U, \varphi) = (U, x^1, \ldots, x^n)$ around p, we might wonder a basis for T_p^*M. By direct calculation, one sees that:

$$(dx^i)_p \left(\frac{\partial}{\partial x^j}(p) \right) = \delta_{ij},$$

and so the set

$$\{(dx^i)_p \mid 1 \leqq i \leqq n\}$$

forms the dual basis to the standard basis on T_pM. As a result, a 1-form α locally can be written as

$$\alpha = \sum_{i=1}^{n} a_i\, dx^i, \qquad a_i \in C^\infty(M).$$

Example 2.3. Consider $M = \mathbb{R}^3$ and the 1-form $\alpha = dz - y\,dx$. We cannot readily give a geometric meaning to the dual space, however we can identify α with its kernel (a sub-bundle of $T\mathbb{R}^3$) which we can certainly draw a picture for! Recall that the kernel of α is

$$\ker \alpha := \{v_p \in T\mathbb{R}^3 \mid \alpha_p(v_p) = 0\}.$$

In particular, $\ker \alpha_p$ is a 2-dimensional subspace of $T_p\mathbb{R}^3$ and hence a plane. So $\ker \alpha_p$ is a plane and hence $\ker \alpha$ is a **plane field**. To visualize this plane field, we consider an arbitrary vector $v_p = a\frac{\partial}{\partial x}(p) + b\frac{\partial}{\partial y}(p) + c\frac{\partial}{\partial z}(p)$ where $p = (p_1, p_2, p_3) \in \mathbb{R}^3$:

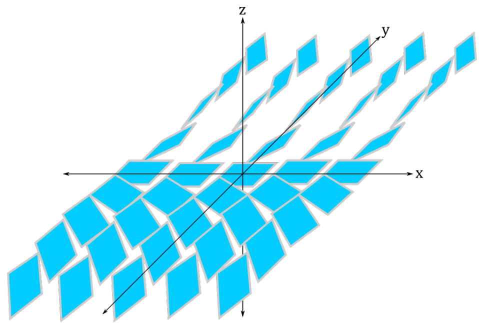
$$\begin{aligned} \alpha_p(v_p) = 0 &\iff (dz(p) - p_2\, dx(p))(v_p) = 0 \\ &\iff dz(p)(v_p) - p_2\, dx(p)(v_p) = 0 \\ &\iff c - p_2 a = 0, \end{aligned}$$

and so it's not hard to see that

$$\ker \alpha_p = \operatorname{span}\left\{ \frac{\partial}{\partial y}(p), \frac{\partial}{\partial x}(p) + p_2 \frac{\partial}{\partial z}(p) \right\}.$$

Along the x, z-plane where $p_2 = 0$, $\ker \alpha_p$ can be identified as the x, y-plane. If we look at the lines in the x, y-plane $p_3 = 0$ which are parallel to the y-axis, $\ker \alpha_p$ rotates (if you are viewing from the origin) counter-clockwise as $p_2 \to \infty$ to get close and closer to the y, z-plane. Below is a picture of the plane field[10].

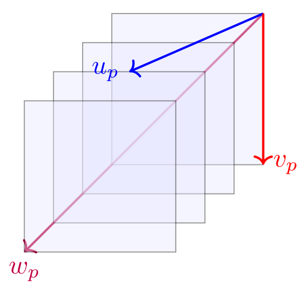
[10] For those who care: we just realized the standard "contact manifold" structure of $\mathbb{R}^3$!

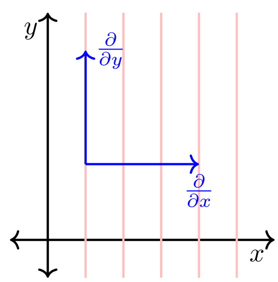


So what *is* the proper geometric interpretation of 1-forms? There is no definitive way, but the popular one (especially with physicists) is as follows. To each point $p \in M$, a 1-form α gives a linear form $\alpha_p := \alpha(p)\colon T_pM \to \mathbb{R}$. For any $x \in \mathbb{R}$ we have that $\alpha_p^{-1}(x)$ (the level-set of α_p at x, if $x = 0$ then this is $\ker \alpha_p$) is a linear subspace of T_pM. Moreover, all level-sets can be imagined to be "parallel" in a way.

The idea is that when we zoom in at $p \in M$ on to α_p, what we see is a little collection of these parallel hyperplanes, stacked like pancakes. Compare this to when you zoom in on a vector field v at p, and we get an arrow v_p.

Now handed v_p, how does one visualize $\alpha_p(v_p)$? Imagine that these level-sets have their values associated to them and that the vector v_p is based at the 0-hyperplane. The vector v_p then pierces the stack of pancakes, and we can "measure" v_p using the number of pancakes pierced as a ruler (pictured below left, $\alpha_p(u_p) = 2, \alpha_p(v_p) = 0, \alpha_p(w_p) = 3$).





In $\mathbb{R}^2$ with the usual coordinates x and y we get that the differential form dx near some point $p \in M$ is imagined as a collection of vertical lines. This way, $\frac{\partial}{\partial y}(p)$ has the obvious property that $(dx)_p \frac{\partial}{\partial y}(p) = 0$ as the vector is parallel to the vertical lines and hence doesn't pierce any (pictured above right). That is, a 1-form can be seen as a coordinate-free way of measuring the intuitive lengths of tangent vectors of a vector field.

2.3 k-forms

We now consider the k^{th} exterior power of T^*M, $\Lambda^k T^*M$ which has a natural vector bundle structure. A **k-form** is a smooth section of $\Lambda^k T^*M$. There is an isomorphism $\Lambda^k T^*M \leftrightarrow (\Lambda TM)^*$ and so each k-form ω evaluates at some $p \in M$ to $\omega_p := \omega(p)\colon \Lambda^k T_pM \to \mathbb{R}$ which can be seen as an alternating multilinear map. The set of all smooth k-forms on M we will denote by $\Omega^k(M)$. The set $\Omega^0(M)$ of 0-forms is identified with $C^\infty(M)$.

Given a k-form α and an ℓ-form β, we can take the exterior product $\alpha \wedge \beta \in \Omega^{k+\ell}$ of the forms (within the exterior algebra) where we characterize the the anticommutativity (alternating) property of the exterior algebra by:

$$\alpha \wedge \beta = (-1)^{k\ell} \beta \wedge \alpha,$$

If $k = \ell = 1$, at a point $p \in M$, the exterior product becomes

$$(\alpha \wedge \beta)_p(v_p, w_p) := \alpha_p(v_p)\beta_p(w_p) - \alpha_p(w_p)\beta_p(v_p), \quad v_p, w_p \in T_pM,$$

when viewed as an alternating map $(\alpha \wedge \beta)_p\colon T_pM \times T_pM \to \mathbb{R}$.

We can extend our geometric interpretation of 1-forms for $k = 2, 3$. Imagine that α, β, ω are 1-forms. We know how to imagine them locally at a point p as a collection of level sets. When taking $\alpha \wedge \beta$, we imagine this as taking the intersection of our level sets. That is, if we had a stack of planes to begin with, now we intersect two stack of planes to get a bundle of lines around p. Our vectors that $(\alpha \wedge \beta)_p$ take as arguments look like $v_p \wedge w_p$ which we can look at as infinitesimal parallelograms (see: cross product as the area of a parallelogram), so evaluated, we count the number of lines in our bundle that this parallelogram passes through to calculate something like an area.

Extending this once more, we get that $(\alpha \wedge \beta \wedge \omega)_p$ appears as an intersection of 3 stacks of planes, which makes a lattice of points. So when we evaluate this 3-form at a $u_p \wedge v_p \wedge w_p$ (something akin to a tiny parallelepiped), we measure the density or volume of a parallelepiped.

2.4 In $\mathbb{R}^n$

Now we turn our attention to $\Omega^k(\mathbb{R}^n)$. In particular, let's let $n = 3$:

$$\begin{aligned}
&\Omega^0(\mathbb{R}^3) = C^\infty(\mathbb{R}^3), \\
&\Omega^1(\mathbb{R}^3) = \{a\,dx + b\,dy + c\,dz \mid a, b, c \in C^\infty(\mathbb{R}^3)\}, \\
&\Omega^2(\mathbb{R}^3) = \{a\,dx \wedge dy + b\,dx \wedge dz + c\,dy \wedge dz \mid a, b, c \in C^\infty(\mathbb{R}^3)\}, \\
&\Omega^3(\mathbb{R}^3) = \{a\,dx \wedge dy \wedge dz \mid a \in C^\infty(\mathbb{R}^3)\}.
\end{aligned}$$

Recall that we have the differential, a map $d\colon \Omega^0(\mathbb{R}^3) \to \Omega^1(\mathbb{R}^3)$. In $\mathbb{R}^3$, this looks like the gradient:

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz \quad \longleftrightarrow \quad \nabla(f) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right).$$

We generalize this (for arbitrary n) to a map $d\colon \Omega^k(\mathbb{R}^n) \to \Omega^{k+1}(\mathbb{R}^n)$:

$$d(f_{i_1\ldots i_k} dx^{i_1} \wedge \ldots \wedge dx^{i_k}) := df_{i_1\ldots i_k} \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_k},$$

called the **exterior derivative**.

Proposition 2.4. *(i) d is $\mathbb{R}$-linear.*

(ii) If $\omega \in \Omega^k(\mathbb{R}^n)$, $\mu \in \Omega^\ell(\mathbb{R}^n)$, then $d(\omega \wedge \mu) = d\omega \wedge \mu + (-1)^k \omega \wedge d\mu$.

(iii) $d(d\omega)) = d^2\omega = 0$.

Consider an $\omega \in \Omega^1(\mathbb{R}^3)$, $\omega = a\,dx + b\,dy + c\,dz$. Then

$$d\omega = \left(\frac{\partial b}{\partial x} - \frac{\partial a}{\partial y}\right) dx \wedge dy + \left(\frac{\partial a}{\partial z} - \frac{\partial c}{\partial x}\right) dz \wedge dx + \left(\frac{\partial c}{\partial y} - \frac{\partial b}{\partial z}\right) dy \wedge dz,$$

which one recognizes immediately (under a certain identification of bivectors $x^i \wedge x^j$ and vectors) is the curl of ω.

Similarly, if $\mu \in \Omega^2(\mathbb{R}^3)$, $\mu = c\,dx \wedge dy + a\,dy \wedge dz + b\,dz \wedge dx$:

$$d\mu = \left(\frac{\partial c}{\partial z} + \frac{\partial a}{\partial x} + \frac{\partial b}{\partial y}\right) dx \wedge dy \wedge dz,$$

which is the divergence of μ!

Finally, from the properties of the exterior derivative, d, we have that $d^2 = 0$ and so the usual theorem about grad/curl/divergence follow that taking any two consecutive operators results in 0 for the following chain:

$$\Omega^0(\mathbb{R}^3) \xrightarrow{\text{grad}} \Omega^1(\mathbb{R}^3) \xrightarrow{\text{curl}} \Omega^2(\mathbb{R}^3) \xrightarrow{\text{div}} \Omega^3(\mathbb{R}^3).$$

The definition for d induces (via charts) an exterior derivative on a general smooth manifold M (with all the same properties), though we won't go into the details about this. The properties listed in Proposition 2.4 apply to the general definition for the exterior derivative.

2.5 Pullbacks, orientations, and the integration of forms

We have seen that when given a smooth $F\colon M \to N$ between manifolds, the tangent space induces a linear map $F_{*,p}\colon T_pM \to T_{F(p)}N$ called the pushforward which "pushes" tangent vectors in T_pM forward to tangent vectors in $T_{F(p)}N$. Because of duality, this gets mirrored in the dual space of T_pM, the cotangent space T_p^*M.

In particular, given a smooth map $F\colon M \to N$ between manifolds, we have an induced map $F^*\colon T^*_{F(p)}N \to T_p^*M$ called the **pullback** of F, defined by:

$$F^*(\alpha_p) := \alpha_p \circ F_{*,p}, \qquad \alpha_p \in T^*_{F(p)}N.$$

This is just the transpose of $F_{*,p}$ and clearly is linear. Some call this the "co-differential" as it is dual to the "differential".

Unique to the cotangent space is the ability to "pullback" 1-forms—it is not true in general[11] that you can pushforward vector fields. The pullback of a 1-form α on M by F is given by:

$$(F^*\alpha)_p(v_p) := \alpha_{F(p)}(F_{*,p}(v_p)), \qquad v_p \in T_pM.$$

[11] Though, if F is a diffeomorphism, you can pushforward vector fields!

This notion of pullback generalizes to k-forms, where we see for $\omega \in \Omega^k(N)$ that:

$$(F^*\omega)_p(v_1, \ldots, v_k) := \omega_{F(p)}(F_{*,p}(v_1), \ldots, F_{*,p}v_k), \qquad v_1, \ldots, v_k \in T_pM.$$

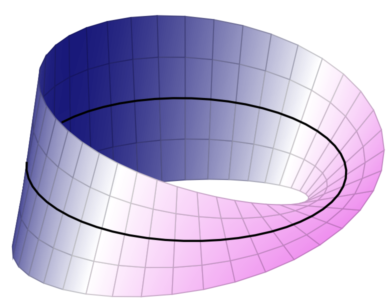
Moreover, this generalized pullback has the following properties for differential forms ω and η:

(i) F^* is $\mathbb{R}$-linear.

(ii) $F^*(\omega \wedge \eta) = (F^*\omega) \wedge (F^*\eta)$.

(iii) $F^*(d\omega) = dF^*(\omega)$.

Before we discuss integration, we have a few restrictions. Firstly, we will only be able to integrate over "oriented" manifolds. That is, things like integration on a Möbius band (pictured below[12]) are out. Secondly, if M is an n-manifold, we will only be able to integrate n-forms. Moreover, these n-forms will need to have what is called **compact support**: zero outside of a compact set[13].



To formalize orientability of a manifold, first consider a finite dimensional vector space V. If we consider all the *ordered* bases of V, then linear maps which take the i^{th} basis element of one basis to the i^{th} of another can be characterized by their determinant—it will either be positive or negative. Fixing a standard basis, we can classify an order basis as either positively oriented (if the corresponding determinant is positive) or negatively oriented (if the corresponding determinant is negative). In this way we define an orientation of V as an additional structure.

Example 2.5. Consider the vector space $\mathbb{R}^3$ with the standard ordered basis (e_1, e_2, e_3). Considering the alternative ordered basis

$$\mathcal{B} = (e_1 + 2e_2, e_3, e_1 - e_2),$$

[12]pgfplot code courtesy of https://tex.stackexchange.com/questions/118563/.

[13]A subset of a topological space is said to be compact if any covering of the subset by open subsets can be refined to a finite covering. This condition is important in order for the integral to be bounded.

the map $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ which is defined by:

$$T(e_1) = e_1 + 2e_2, \quad T(e_2) = e_3, \quad T(e_3) = e_1 - e_2,$$

has the corresponding matrix

$$T = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix},$$

which has determinant $\det T = 3 > 0$ and so $\mathcal{B}$ is positively oriented with respect to the standard basis.

Returning to manifolds, if M is a manifold, then an orientation on M consists of a choice of orientations on the tangent spaces T_pM for each p which are positively oriented relative to those of nearby, overlapping charts. This is captured by the Jacobian matrix of the transition maps.

Formally, an n-manifold M is **orientable** if it admits an **oriented atlas**: an atlas such that any transition map of overlapping charts has an everywhere positive Jacobian determinant on the intersection.

If we consider an n-form ω, then $\omega_p\colon T_pM \to \mathbb{R}$ has the property that for any ordered bases $(v_1, \ldots, v_n)$ and $(w_1, \ldots, w_n)$ of T_pM, the map $T\colon T_pM \to T_pM$ defined by $T(v_i) = w_i$ satisfies:

$$\omega_p(w_1, \ldots, w_n) = \det(T) \cdot \omega_p(v_1, \ldots, v_n).$$

This determines an orientation for these tangent spaces. Given that ω is smooth (or simply continuous), the existence of such an n-form that never vanishes is equivalent to having an oriented atlas:

Proposition 2.6. *An n-manifold M is orientable if and only if it admits a never-vanishing (smooth) n-form, called a* ***volume form****.*

Now we can discuss integration on manifolds. Let M be an orientable n-manifold, and let $\omega \in \Omega^n(M)$ have compact support in a chart (U, φ) on M. Then the integral of ω on U is

$$\int_U \omega := \int_{\varphi(U)} (\varphi^{-1})^* \omega,$$

which is an integral of an n-form on $\varphi(U)$ and has the form of $\int_{\varphi(U)} f(x)\, dx^1 \wedge \ldots \wedge dx^n$. It is integrated as $\int_{\varphi(U)} f(x)\, dx^1 \ldots dx^n$, your typical Riemann integral.

Example 2.7 (Tu, 22.7)**.** Consider spherical coordinates. Let r denote the distance of a point in $\mathbb{R}^3$ from the origin, ϕ be the polar angle from the positive z-axis, and θ be the azimuthal angle from the positive x-axis. Note that the sphere (embedded in $\mathbb{R}^3$) has the volume form

$$\omega = x\, dy \wedge dz - y\, dx \wedge dz + z\, dx \wedge dy,$$

and hence is orientable. Consider

$$U = \{(\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi) \in S^2 \mid 0 < \phi < \pi, 0 < \theta < 2\pi\},$$

which constitutes a chart (U, ϕ, θ) on S^2. We wish to calculate $\int_U \sin\phi \, d\phi \wedge d\theta$. To this end, let $f = (\phi, \theta)$ be the coordinate map on U and define:

$$u := (f^{-1})^*\phi = \phi \circ f^{-1}, \qquad v := (f^{-1})^*\theta = \theta \circ f^{-1}.$$

These correspond to the coordinate functions on

$$f(U) = \{(u, v) \in \mathbb{R}^2 \mid 0 < u < \pi, 0 < v < 2\pi\}.$$

Now we just compute:

$$\begin{aligned}
\int_U \sin\phi \, d\phi \wedge d\theta &= \int_{f(U)} (f^{-1})^*(\sin\phi \, d\phi \wedge d\theta) \\
&= \int_{f(U)} \sin(\phi \circ f^{-1}) \, d(f^{-1})^*\phi \wedge d(f^{-1})^*\theta \\
&= \int_{f(U)} \sin u \, du \wedge dv \\
&= \int_{f(U)} \sin u \, du \, dv \\
&= \int_0^{2\pi} \int_0^{\pi} \sin u \, du \, dv \\
&= 2\pi \int_0^{\pi} \cos u \, du \\
&= 4\pi.
\end{aligned}$$

So far all our manifolds have been "open" in the sense that they don't typically have an "edge". The definition for a manifold can be extended to include **manifolds with boundary** by requiring that any open subset of M be homeomorphic to an open subset of $\mathbb{H}^n$,

$$\mathbb{H}^n := \{(x^1, \ldots, x^n) \in \mathbb{R}^n \mid x^n \geqq 0\},$$

the **Euclidean half-space**. In this way, some points $p \in M$ are mapped by charts to points in $\mathbb{H}^n$ with $x^n = 0$. We call these points **boundary points**, and denote the set of them by ∂M, the **boundary** of M. If M is an n-manifold, then ∂M constitutes an $(n-1)$-manifold (by using the restriction of charts to $\mathbb{R}^{n-1} \subseteq \mathbb{H}^n$). The theory of manifolds with boundary is developed analogously to that of manifolds without boundary. We finish off this integration by stating a general formulation of a famous theorem from multivariable calculus (and of course, without proof!).

Theorem 2.8 (Stokes')**.** *Let M be an orientable n-manifold with boundary. If ω is an $(n-1)$-form with compact support on M, then*

$$\int_M d\omega = \int_{\partial M} \omega.$$

2.6 Cohomology

In algebra, a **cochain** of vector spaces is a family $\{C^k\}_{k\in\mathbb{Z}}$ of vector spaces,

$$\cdots \xrightarrow{d^{-1}} C^0 \xrightarrow{d^0} C^1 \xrightarrow{d^1} C^2 \xrightarrow{d^2} \cdots$$

where $\{d^k : C^k \to C^{k+1}\}_{k\in\mathbb{Z}}$ is a matching family of linear maps satisfying

$$\operatorname{im} d^k \subseteq \ker d^{k+1}.$$

We denote the cochain by $C^\bullet$. If instead of $\subseteq$ we have equality, then the cochain is said to be **exact**. Because the C^k are vector spaces we can take the quotients,

$$H^k(C^\bullet) := \ker d^{k+1} / \operatorname{im} d^k,$$

which we call the cohomology groups (they are vector spaces which are "abelian groups" under addition). If $C^\bullet$ is exact, then we notice that the $H^k(C^\bullet)$ are trivial, and so we tend to view the homology groups as measuring to what degree $C^\bullet$ fails to be exact.

Now define the **de Rham complex** $\Omega(M)$ of M:

$$\Omega(M) := \bigoplus_{i=0}^{\dim M} \Omega^k(M),$$

which is a graded algebra. Since $d^2 = 0$, we have that

$$d(\Omega^{k-1}(M)) \subseteq \ker d|_{\Omega^k(M)}, \tag{$\star$}$$

which gives us a cochain (implicitly, every space preceding $\Omega^0(M)$ is $\{0\}$)

$$\Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{d} \Omega^3(M) \xrightarrow{d} \ldots$$

where taking any two successive arrows is 0 as a result of ($\star$). Define $Z^k(M) := \ker d|_{\Omega^k(M)}$ as the set of **closed k-forms** (that is, forms $\omega \in \Omega^k(M)$ such that $d\omega = 0$), and $B^k(M) := d(\Omega^{k-1}(M))$ as the set of **exact k-forms** (that is, forms $\omega \in \Omega^k(M)$ such that there exists a $(k-1)$-form μ such that $d\mu = \omega$). In other words, ($\star$) says that $B^k(M) \subseteq Z^k(M)$ and so we can take the quotient

$$H^k_{dR}(M) := Z^k(M)/B^k(M)$$

which measures to what degree closed forms fail to be exact forms. That is, we say two k-forms are "cohomologous" (in the same equivalence class) if they differ by an exact k-form. In particular, exact forms are cohomologous to 0. Because the de Rham complex $\Omega(M)$ has a product structure given by the wedge product, $H_{dR}(M)$ inherits this product structure, given by:

$$[\omega] \wedge [\eta] = [\omega \wedge \eta],$$

and so $H_{dR}(M) := \bigoplus_{k=0}^n H^k_{dR}(M)$ forms a graded-commutative algebra, called the **de Rham cohomology of** M. The significance of the de Rham cohomology is that it is an algebraic invariant:

Theorem 2.9. *Suppose that M and N are diffeomorphic smooth manifolds. Then $H_{dR}(M) \cong H_{dR}(N)$ as vector spaces (and hence have the same dimension).*

Differential topology is often summarized as the use of topology in order to determine manifolds up to diffeomorphism, and so with respect to this, the de Rham cohomology of M serves as an important tool to this end. The dimension $b_k(M) := \dim H^k_{dR}(M)$ is called the k^{th} **Betti number** of M, where we note by a dimension argument on $\Omega^k(M)$ that for $k > \dim M$, $b_k(M) = 0$. Using the Betti numbers, we can calculate the **Euler characteristic** of M as the (finite) alternating sum:

$$\chi(M) := b_0(M) - b_1(M) + b_2(M) - b_3(M) + \dots,$$

another (rather famous) algebraic topological invariant that is commonly used in many areas of mathematics. In the context of polyhedra, the Euler characteristic is $\chi = V - E + F$ where V, E, and F are respectively the number of vertices, edges, and faces of the polyhedron in question. It's well known that any convex polyhedron has an Euler characteristic of 2.

Example 2.10 (de Rham cohomology of S^1)**.** First we consider $H^0_{dR}(S^1)$ and then finally $H^1_{dR}(S^1)$. We know that $H^k_{dR}(S^1) = \{0\}$ for $k > \dim S^1 = 1$.

(i) $H^0_{dR}(S^1)$: A 0-form f is closed if $df = 0$, so locally f is constant. Since S^1 is connected, if it is locally constant, it is globally constant on S^1. Hence $H^1_{dR}(S^1) \cong \mathbb{R}$.

(ii) $H^1_{dR}(S^1)$: First note that any 1-form on S^1 is closed (otherwise $\Omega^2(S^1)$ is non-zero) and so $H^1_{dR}(S^1) = Z^1(S^1)$. Consider the maps:

$$\mathbb{R} \xrightarrow{e} S^1 \xhookrightarrow{i} \mathbb{R}^2,$$

where $e(t) = (\cos t, \sin t)$ and i is the inclusion. Consider the non-vanishing 1-form $\tilde{\alpha} := -y\,dx + x\,dy$ on $\mathbb{R}^2$ which is closed (exercise) but not exact[14]. Pull back this form along i to a 1-form $\alpha := i^*\tilde{\alpha}$ on S^1.

One can calculate that

$$i_* e_* \frac{d}{dt} = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y},$$

and so:

$$1 = \tilde{\alpha}\left(i_* e_* \frac{d}{dt}\right) = i^*\tilde{\alpha}\left(e_* \frac{d}{dt}\right) = \alpha\left(e_* \frac{d}{dt}\right) = e^*\alpha\left(\frac{d}{dt}\right),$$

where we conclude that $e^*\alpha = dt$.

[14]If there were such an $f \in C^\infty(\mathbb{R}^2)$ such that $df = \tilde{\alpha}$, then f would attain a maximum/minimum on a compact subset such as S^1. In this case, df would vanish, however it is non-vanishing on S^1.

The map $f \mapsto e^*f := f \circ e$ is an isomorphism of $C^\infty(S^1)$ to the set of smooth, 2π-periodic functions on $\mathbb{R}$. The map $e^*\colon \Omega^1(S^1) \to \Omega^1(\mathbb{R})$ has for each $\beta \in \Omega^1(S^1)$ that $e^*\beta = g\,dt$ for some $g \in C^\infty(\mathbb{R})$ which is 2π-periodic. So there exists a $f \in C^\infty(S^1)$ with $g = e^*f$ so that $e^*\beta = g\,dt = e^*f\,e^*\alpha = e^*(f\alpha)$. So e^* gives us a correspondence $\beta \leftrightarrow f\alpha$.

Define a new map:

$$\phi\colon \Omega^1(S^1) \to \mathbb{R}\colon \beta \mapsto \int_{S^1} \beta.$$

This map ϕ is surjective onto $\mathbb{R}$ because

$$\phi(\alpha) = \int_{S^1} \alpha = \int_{[0,2\pi]} e^*\alpha = \int_0^{2\pi} dt = 2\pi \neq 0.$$

Hence by linearity, it must be surjective onto $\mathbb{R}$. Stokes' theorem says that any exact form is in the kernel of ϕ. On the other hand, suppose that $f\alpha \in \ker\phi$. Then $\int_{S^1} f\alpha = 0$ and so $\tilde{g}(t) := \int_0^t e^*f$ is a 2π-periodic function and hence is identified with some $g \in C^\infty(S^1)$ so that $\tilde{g} = e^*g$. We note that

$$e^*(dg) = d(e^*g) = d\tilde{g} = e^*f\,dt = e^*fe^*\alpha = e^*(f\alpha),$$

and hence (since e^* is an isomorphism), $dg = f\alpha$ is exact.

Hence by the first isomorphism theorem[15]:

$$\Omega^1(S^1)/\ker\phi = H^1_{dR}(S^1) \cong \mathbb{R}.$$

In conclusion, we have found that:

$$H^k_{dR}(S^1) = \begin{cases} \mathbb{R}, & k = 0, 1; \\ 0, & k \geqq 2, \end{cases}$$

and the Euler characteristic of S^1 is $\chi(S^1) = 1 - 1 = 0$.

Intuitively, the Betti numbers give the number of k-dimensional "holes" in the manifold. So the common explanation is that $b_0(M)$ counts the connected components, $b_1(M)$ counts the number of "circular" holes, and $b_2(M)$ counts the number of hollow cavities. In our example of S^1, we have one connected component and one circular hole. As one can see, actually computing the de Rham cohomology is non-trivial, and because of this, it is rare that the entirety of the cohomology of a manifold M is calculated. It is only in special cases is it easily done.

[15]If $T\colon V \to W$ is a linear map of vector spaces, then the first isomorphism theorem says that $V/\ker T \cong \operatorname{im} T$.

A Further Reading

For more on differential geometry, see:

- *Introduction to manifolds*, Loring W. Tu.
- *Differential Manifolds*, Antoni A. Kosinski.
- *An Introduction to Differential Manifolds*, Jacques Lafontaine.

For more on differential geometry and applications to algebraic topology, see:

- *Differential Forms in Algebraic Topology*, Raoul Bott & Loring W. Tu.

For more on differential geometry as it relates to physics, see:

- *Gauge Fields, Knots and Gravity*, John Baez & Javier P. Muniain.