A little about Gram-Schmidt

Luke Volk

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In an inner product space

Theorem 1 (Gram-Schmidt)**.** *If* $(V, \langle\cdot,\cdot\rangle)$ *is a finite dimensional inner product space has a basis* $(u_1, u_2, \ldots, u_n)$ *such that:*

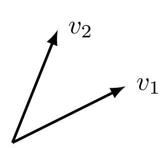
$$\langle u_i, u_j \rangle = \delta_{ij}.$$

The proof is well known. Take a basis $(v_1, v_2, \ldots, v_n)$ (which doesn't necessarily satisfy the property of Theorem 1), and then define:

$$u_1 := v_1,$$
$$u_i := v_i - \sum_{j=1}^{i-1} \frac{\langle v_i, u_j \rangle}{\langle u_j, u_j \rangle} u_j, \qquad 2 \leqq i \leqq n,$$

and then note we may easily normalize all the vectors. This (after doing your homework) works well.

Elaboration . *It is instructive to think about this process in* $\mathbb{R}^2$*. If I hand you a basis consisting of vectors* $v_1, v_2 \in \mathbb{R}^2$*, then in general we are going to have something that looks like this:*



If we take $u_1 := v_1$*, then we can project* v_2 *onto* u_1 *(using the formula* $proj_{u_1}(v_2) = \frac{\langle v_2, u_1 \rangle}{\langle u_1, u_1 \rangle} u_1$*), and then subtract this projection from* v_2 *to yield a vector*

$$u_2 := v_2 - proj_{u_1}(v_2),$$

> *which is orthogonal to u_1!*
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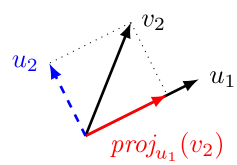
> 
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> *By subtracting the projection $proj_{u_1}(v_2)$ from v_2, we have removed the part of v_2 which is not orthogonal to u_1.*

However, we can try phrasing the proof in a different way:

1. Pick any unit vector $u_1 \in V$. Then consider $V_1 := \text{span}\{u_1\}$ and the orthogonal complement:
$$V_1^\perp := \{v \in V \mid \forall u \in V_1 \,.\, \langle v, u\rangle = 0\}.$$
2. If $\dim V_1^\perp = 0$, then we have our basis. Otherwise we have a non-zero $v_2 \in V_1^\perp$ (such that $\langle u_1, v_2\rangle = 0$), and we normalize it to u_2 which satisfies the analogous property.
3. Set $V_2 := \text{span}\{u_1, u_2\}$ and analogously define $V_2^\perp$.
4. Continue this process until $\dim V_n^\perp = 0$ for some n.

The other (more constructive) proof of Gram-Schmidt has the benefit of showing how we can actually choose the u_i, given that we have a basis consisting of a bunch of v_i.

In a symplectic space

Now let V be a finite dimensional vector space and $\omega\colon V \times V \to \mathbb{R}$ a non-degenerate bilinear form. Note that our inner product from before was a non-degenerate bilinear form. Note we can follow a similar process:

1. Pick any vector $u_1 \in V$. Then consider $V_1 := \text{span}\{u_1\}$, defining:
$$V_1^\omega := \{v \in V \mid \forall u \in V_1 \,.\, \omega(v, u) = 0\}.$$

Note that before, our inner product was symmetric, while ω is not necessarily symmetric. This makes the ω-orthogonal complement, V_1^ω, the **left ω-orthogonal complement** of V_1. Similarly, you get a **right** complement. If we have that ω is skew-symmetric (i.e. $\omega(u, v) = -\omega(v, u)$) or symmetric, then the left and right complements coincide (exercise: when else do they coincide?). We will now restrict our attention to when ω is skew-symmetric. Now note that we run into a problem if we continued following the above.

Example 2. Consider $\mathbb{R}^2$ with the skew-symmetric, non-degenerate bilinear form ω given by:

$$\omega(u, v) := u^T \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} v.$$

If we just pick $u_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, then we note that:

$$V_1^\omega = V_1.$$

This is problematic. We can't pick a vector $u_2 \in V_1^\omega$ such that $\{u_1, u_2\}$ is linearly independent, unlike with the inner product (exercise: explain why this happens with skew-symmetric forms). But what we *can* do, by non-degeneracy of ω, is find a $v_1 \in V$ such that $\omega(u_1, v_1) = 1$ (exercise: find one for this example).

So how we proceed in coming up with a more or less standardized basis is by doing just that:

2. Find $v_1 \in V$ such that $\omega(u_1, v_1) = 1$ (guaranteed by non-degeneracy of ω).

It's not clear where to go from here, so let's see if we at all fixed our problem from before (as now our algorithm solves Example 2):

3. Define $W_1 := \text{span}\{u_1, v_1\}$ and then:

$$W_1^\omega = \{v \in V \mid \forall w \in W_1 \,.\, \omega(v, w) = 0\}.$$

The thing that was nice with the inner product was that for any subspace $U \leq V$, we have (exercise!) that:

$$V = U \oplus U^\perp.$$

In the above example, we found that this is not always the case with a skew-symmetric, non-degenerate bilinear form ω. However, if we want to use the same approach to defining a nice basis (nice with respect to ω), then we would like this to at least be true in the circumstance for W_1...maybe?

Proposition 3. *With the notation as above, we have:*

$$V = W_1 \oplus W_1^\omega.$$

Proof. Below are some hints of the exercise in linear algebra:

- You want to prove that the intersection of the summands is $\{0\}$ and that $V = W_1 + W_1^\omega$.

- Take an arbitrary vector w in the intersection $W_1 \cap W_1^\omega$: what is $\omega(u_1, w)$? $\omega(v_1, w)$?
- If $w \in V$, then add and subtract both $\omega(w, u_1)v_1$ and $\omega(w, v_1)u_1$ to w and re-arrange the expression in a smart way to conclude that it is clear that $w \in W_1 + W_1^\omega$.

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Thus we have exactly what we want to proceed as we did with the inner product space! Keep in mind now that out of each W_i we will take not one vector, but two!

4. Take a non-zero $u_2 \in W_1^\omega$ and find $v_2 \in W_1^\omega$ such that $\omega(u_2, v_2) = 1$. Define $W_2 := \text{span}\{u_2, v_2\}$ and then W_2^ω analogously.

In this manner, we can build up a direct sum of vector subspaces:

$$V = W_1 \oplus W_2 \oplus \dots$$

which, by finite dimensionality of V, must come to a stop. Hence we get a special basis:

Theorem 4 (Symplectic Gram-Schmidt)**.** *Let V be a finite dimensional vector space and ω a skew-symmetric, non-degenerate bilinear form on V[a]. Then there exists a basis $(u_1, v_1, u_2, v_2, \dots, u_n, v_n)$ such that:*

$$\begin{cases} \omega(u_i, u_j) = \omega(v_i, v_j) = 0, \\ \omega(u_i, v_j) = \delta_{ij}. \end{cases}$$

[a] A skew-symmetric, non-degenerate bilinear form ω on V is called a **symplectic form**. The pair (V, ω) is called a **symplectic vector space**.

We conclude this mathematical adventure with the following corollaries:

Corollary 5. *Note that:*

(i) Such a V is necessarily even dimensional, $\dim V = 2n$.

(ii) Under this basis, ω is of the form:

$$\omega = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix},$$

where I is the $n \times n$ identity matrix.